ON THE MARTINGALE PROBLEM FOR DEGENERATE-PARABOLIC PARTIAL DIFFERENTIAL OPERATORS WITH UNBOUNDED COEFFICIENTS AND A MIMICKING THEOREM FOR ITÔ PROCESSES

PAUL M. N. FEEHAN AND CAMELIA POP

ABSTRACT. Using results from our companion article [21] on a Schauder approach to existence of solutions to a degenerate-parabolic partial differential equation, we solve three intertwined problems, motivated by probability theory and mathematical finance, concerning degenerate diffusion processes. We show that the martingale problem associated with a degenerate-elliptic differential operator with unbounded, locally Hölder continuous coefficients on a half-space is well-posed in the sense of Stroock and Varadhan. Second, we prove existence, uniqueness, and the strong Markov property for weak solutions to a stochastic differential equation with degenerate diffusion and unbounded coefficients with suitable Hölder continuity properties. Third, for an Itô process with degenerate diffusion and unbounded but appropriately regular coefficients, we prove existence of a strong Markov process, unique in the sense of probability law, whose one-dimensional marginal probability distributions match those of the given Itô process.

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1. Introduction

Consider a time-dependent, degenerate-elliptic differential operator defined by *unbounded* coefficients (a,b) on the half-space $\mathbb{H} := \mathbb{R}^{d-1}\times(0,\infty)$ with $d\geq 2$,

$$\mathscr{A}_t v(x) := \frac{1}{2}\sum_{i,j=1}^d x_d a_{ij}(t,x) v_{x_i x_j}(x) + \sum_{i=1}^d b_i(t,x) v_{x_i}(x), \quad (t,x)\in[0,\infty)\times\mathbb{H}, \tag{1.1}$$

and $a=(a_{ij})$, $b=(b_i)$, and $v\in C^2(\overline{\mathbb{H}})$. The operator $\mathscr{A}_t$ becomes *degenerate* along the boundary $\partial\mathbb{H}=\{x_d=0\}$ of the half-space. In this article, motivated by applications to probability theory and mathematical finance [2, 17, 23, 34], we apply the main result of our companion article[1] [21] to solve three intertwined problems concerning degenerate diffusion processes related to (1.1).

First, we show that the martingale problem §1.1.1 for the degenerate-elliptic operator with unbounded coefficients, $\mathscr{A}_t$, in (1.1) is well-posed in the sense of Stroock and Varadhan [38]. Second, as discussed in more detail in §1.1.2, we prove existence, uniqueness, and the strong Markov property for weak solutions, $\widehat{X}$, to a degenerate stochastic differential equation with unbounded coefficients,

$$\begin{aligned} d\widehat{X}(t) &= b(t,\widehat{X}(t))dt + \sigma(t,\widehat{X}(t))d\widehat{W}(t), \quad t\geq s,\\ \widehat{X}(s) &= x. \end{aligned} \tag{1.2}$$

when the coefficient σ is a square root of the coefficient matrix $x_d a$ in $\mathscr{A}_t$ in (1.1), that is, when $\sigma\sigma^* = x_d a$ on $\mathbb{H}_T$, where $\mathbb{H}_T := (0,T)\times\mathbb{H}$ is the open half-cylinder with $0<T<\infty$. Third, suppose we are given a degenerate Itô process, X, with unbounded coefficients,

$$\begin{aligned} dX(t) &= \beta(t)dt + \xi(t)dW(t), \quad t\geq 0,\\ X(0) &= x, \end{aligned} \tag{1.3}$$

whose coefficients (ξ,β) are related to those of (1.2) as explained in §1.1.3. When the coefficients (b,σ) in (1.2) are determined by the coefficients (ξ,β) in (1.3) as described in §1.1.3, we show that the weak solution $\widehat{X}$ to (1.2) "mimics" the Itô process (1.3) in the sense that $\widehat{X}(t)$ has the same one-dimensional marginal probability distributions as $X(t)$, for all $t\geq 0$ if $\widehat{X}(0)=X(0)\in\overline{\mathbb{H}}$. Our mimicking theorem complements that of Gyöngy [23], who assumes that (1.2) is non-degenerate with bounded, measurable coefficients, that of Brunick and Shreve [11, 13], who allow (1.2) to

[1] Our longer previous manuscript [20] combined [21] with the present article.

be degenerate with unbounded, measurable coefficients, and those of Bentata and Cont [9] and Shi and Wang [37, 39] who prove mimicking theorems for a discontinuous semimartingale process with a non-degenerate diffusion component and bounded coefficients.

1.1. **Summary of main results.** We describe our results outlined in the preamble to §1.

1.1.1. *Existence and uniqueness of solutions to the martingale problem for a degenerate-parabolic operator with unbounded coefficients.* We define an analogue of the classical martingale problem ([38, p. 138], [26, Definition 5.4.5 & 5.4.10]) when $\mathbb{R}^d$ is replaced by the half-space $\mathbb{H}$.

For $x, y \in \mathbb{R}$, we denote $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$, $x^+ = \max\{x, 0\}$, and $x^- = \min\{x, 0\}$. The space $C_{\text{loc}}([0,\infty);\overline{\mathbb{H}})$ of continuous functions, $u : [0,\infty) \to \overline{\mathbb{H}}$, endowed with the topology of uniform convergence on compact sets is a complete, separable, metric space. We denote by $\mathscr{B}(C_{\text{loc}}([0,\infty);\overline{\mathbb{H}}))$ the Borel σ-algebra induced by this topology. As in [26, Problem 2.4.2], we see that $\mathscr{B}(C_{\text{loc}}([0,\infty);\overline{\mathbb{H}}))$ is also the σ-algebra generated by the cylinder sets (1.4). Following [26, Problem 2.4.2, Equation (5.3.19) & Remark 5.4.16], we consider the filtration $\{\mathscr{B}_t(C_{\text{loc}}([0,\infty);\overline{\mathbb{H}}))\}_{t\geq 0}$ given by

$$\mathscr{B}_t(C_{\text{loc}}([0,\infty);\overline{\mathbb{H}})) := \varphi_t\left(\mathscr{B}(C_{\text{loc}}([0,\infty);\overline{\mathbb{H}}))\right), \quad \forall t \geq 0, \tag{1.4}$$

where $\varphi_t : C_{\text{loc}}([0,\infty);\overline{\mathbb{H}}) \to C_{\text{loc}}([0,\infty);\overline{\mathbb{H}})$ is defined by

$$\varphi_t(\omega) := \omega(t \wedge \cdot), \quad \forall \omega \in C_{\text{loc}}([0,\infty);\overline{\mathbb{H}}).$$

Definition 1.1 (Solution to a martingale problem for an operator on a half-space)**.** Given $(s, x) \in [0,\infty) \times \overline{\mathbb{H}}$, a probability measure $\widehat{\mathbb{P}}^{s,x}$ on

$$(C_{\text{loc}}([0,\infty);\overline{\mathbb{H}}), \mathscr{B}(C_{\text{loc}}([0,\infty);\overline{\mathbb{H}}))$$

is a *solution to the martingale problem associated to* $\mathscr{A}_t$ *in* (1.1) *starting from* (s, x) if

$$M_t^v(\omega) := v(\omega(t)) - v(\omega(s)) - \int_s^t \mathscr{A}_u v(\omega(u))\, du, \quad t \geq s,\ \omega \in C_{\text{loc}}([0,\infty);\overline{\mathbb{H}}),$$

is a continuous $\widehat{\mathbb{P}}^{s,x}$-martingale, for every $v \in C_0^2(\overline{\mathbb{H}})$, with respect to the filtration $\widehat{\mathscr{F}_t} = \mathscr{G}_{t+}$, where $\mathscr{G}_t$ is the augmentation under $\widehat{\mathbb{P}}^{s,x}$ of the filtration $\{\mathscr{B}_t(C_{\text{loc}}([0,\infty);\overline{\mathbb{H}})\}_{t\geq 0}$, and

$$\widehat{\mathbb{P}}^{s,x}\left(\omega \in C_{\text{loc}}([0,\infty);\overline{\mathbb{H}}) : \omega(t) = x, 0 \leq t \leq s\right) = 1. \tag{1.5}$$

□

Remark 1.2 (Reduction to usual filtration). [26, Remark 5.4.16] By modifying the statement and solution to [26, Problem 5.4.13] (that is, replacing $\mathbb{R}^d$ by $\mathbb{H}$), we see that if M_t^v is a martingale with respect to the filtration $\{\mathscr{B}_t(C_{\text{loc}}([0,\infty);\overline{\mathbb{H}})\}_{t\geq 0}$, then it is a martingale with respect to the enlarged filtration $\widehat{\mathscr{F}_t}$.

Theorem 1.3 (Existence and uniqueness of solutions to the martingale problem for a degenerate-elliptic operator with unbounded coefficients)**.** *Suppose the coefficients* (a, b) *in* (1.1) *obey the conditions in Assumption 2.2. Then, for any* $(s, x) \in [0,\infty) \times \overline{\mathbb{H}}$*, there is a unique solution,* $\widehat{\mathbb{P}}^{s,x}$*, to the martingale problem associated to* $\mathscr{A}_t$ *in* (1.1) *starting from* (s, x)*.*

When the initial condition (s, x) is clear from the context, we write $\widehat{\mathbb{P}}$ instead of $\widehat{\mathbb{P}}^{s,x}$. For brevity, when the initial condition is $(0, x)$, we sometimes write $\widehat{\mathbb{P}}^x$ instead of $\widehat{\mathbb{P}}^{0,x}$.

Remark 1.4 (Well-posedness of the classical martingale problem in [38])). Standard results which ensure *existence* of solutions to the classical martingale problem associated with

$$\tilde{\mathscr{A}_t} v(t,x) := \frac{1}{2}\sum_{i,j=1}^d \tilde{a}_{ij}(t,x) v_{x_i x_j}(x) + \sum_{i=1}^d \tilde{b}_i(t,x) v_{x_i}(x), \quad (t,x) \in [0,\infty)\times\mathbb{R}^d, \tag{1.6}$$

require that the coefficients

$$\begin{aligned} \tilde{a} &: [0,\infty)\times\mathbb{R}^d \to \mathbb{S}_d, \\ \tilde{b} &: [0,\infty)\times\mathbb{R}^d \to \mathbb{R}^d. \end{aligned} \tag{1.7}$$

be bounded and continuous [26, Theorem 5.4.22], [38, Theorem 6.1.7]; here, $\mathbb{S}_d \subset \mathbb{R}^{d\times d}$ denotes the closed, convex subset of *non-negative* definite, symmetric matrices and $\tilde{a} = (\tilde{a}_{ij})$, $\tilde{b} = (\tilde{b}_i)$. Standard results which ensure *uniqueness* of solutions require, in addition, that the coefficients $(\tilde{a},\tilde{b})$ are Hölder continuous and that the matrix a is uniformly elliptic (see [26, Theorem 5.4.28, Corollary 5.4.29, and Remark 5.4.29] for the time-homogeneous martingale problem). Strict ellipticity of the second order coefficients matrix is required for the uniqueness of the martingale problem to hold, as in [38, Theorem 7.2.1].

Remark 1.5 (Approaches to proving uniqueness in the classical martingale problem). *Uniqueness* of solutions to the classical martingale problem is often shown [26, §5.4] by proving *existence* of solutions in $C([0,T]\times\mathbb{R}^d)\cap C^{1,2}((0,T)\times\mathbb{R}^d)$ to the terminal value problem for the parabolic partial differential equation,

$$\begin{cases} u_t + \tilde{\mathscr{A}_t} u = 0 & \text{on } (0,T)\times\mathbb{R}^d, \\ u(T,\cdot) = g & \text{on } \mathbb{R}^d, \end{cases}$$

where $g \in C_0^\infty(\mathbb{R}^d)$ and $\tilde{\mathscr{A}_t}$ is given by (1.6).

Remark 1.6 (Comments on uniqueness). While [26, Remark 5.4.31] might appear to provide a simple solution to the uniqueness property asserted by Theorem 1.3 when the nonnegative definite matrix-valued function $x_d a$ is in $C^2(\mathbb{H};\mathbb{S}_d)$, that is not the case. Although we might extend the coefficient, $x_d a$, as a nonnegative definite matrix-valued function $x_d^+ a$ or $|x_d| a$ in $C^{0,1}(\mathbb{R}^d;\mathbb{S}_d)$, such extensions are not in $C^2(\mathbb{R}^d;\mathbb{S}_d)$, as required by [26, Remark 5.4.31].

1.1.2. *Existence and uniqueness of weak solutions to a degenerate stochastic differential equation with unbounded coefficients.* Given a function,

$$\bar{a} : [0,\infty)\times\overline{\mathbb{H}} \to \mathbb{S}_d,$$

then $\bar{a}(t,x)$ is a non-negative definite, symmetric, real matrix for each $(t,x)\in[0,\infty)\times\overline{\mathbb{H}}$, and so there is a function

$$\sigma : [0,\infty)\times\overline{\mathbb{H}} \to \mathbb{R}^{d\times d}, \tag{1.8}$$

such that

$$\bar{a}(t,x) = \sigma(t,x)\sigma^*(t,x), \quad \forall (t,x)\in[0,\infty)\times\overline{\mathbb{H}}. \tag{1.9}$$

By [22, Lemma 6.1.1], we may choose $\sigma \in C_{\mathrm{loc}}([0,\infty)\times\overline{\mathbb{H}};\mathbb{R}^{d\times d})$ when $\bar{a}\in C_{\mathrm{loc}}([0,\infty)\times\overline{\mathbb{H}};\mathbb{S}_d)$; this continuity property is guaranteed by the conditions on $\bar{a}$ in (2.16) and (2.18) implied through (1.10).

The coefficient functions (σ,b) define a degenerate stochastic differential equation (1.2). Let $\mathbb{S}_d^+ \subset \mathbb{R}^{d\times d}$ denote the convex subset of *positive* definite, symmetric matrices. Unless other conditions are explicitly substituted, we require in this article that the coefficients (σ,b) satisfy

Assumption 1.7 (Properties of the coefficients of the stochastic differential equation)**.** The coefficient functions (σ, b) in (1.2) obey the following conditions.

(1) There is a function $a : [0,\infty) \times \overline{\mathbb{H}} \to \mathbb{S}^d_+$ such that

$$\bar{a}(t,x) = x_d^+ a(t,x), \quad \forall (t,x) \in [0,\infty) \times \overline{\mathbb{H}}. \tag{1.10}$$

(2) The coefficient functions (a, b) obey the conditions in Assumption 2.2.

The constraints on the coefficients (σ, b) in Assumption 1.7 are mild enough that they include many examples of interest in mathematical finance.

Example 1.8 (Parabolic Heston partial differential equation)**.** The conditions in Assumption 2.2 are obeyed by the coefficients of the parabolic Heston partial differential operator,

$$-Lu = -u_t + \frac{y}{2}\left(u_{xx} + 2\varrho\sigma u_{xy} + \sigma^2 u_{yy}\right) + (r - q - y/2)u_x + \kappa(\theta - y)u_y - ru, \tag{1.11}$$

where $q \geq 0, r \geq 0, \kappa > 0, \theta > 0, \sigma > 0$, and $\varrho \in (-1,1)$ are constants.

Example 1.9 (Heston stochastic differential equation)**.** The conditions in Assumption 1.7 are obeyed by the coefficients of the $\mathbb{R}^2$-valued log-Heston process [24] with killing,

$$\begin{aligned} dX_1(t) &= (r - q - X_2(t)/2)dt + \sqrt{X_2(t)}dW_1(t), \\ dX_2(t) &= \kappa(\theta - X_2(t))dt + \sigma\sqrt{X_2(t)}\left(\varrho dW_1(t) + \sqrt{1-\varrho^2}dW_2(t)\right), \end{aligned} \tag{1.12}$$

where the coefficients are as in Example 1.8.

The following theorem does not follow by the classical results [19, Theorem 5.3.3, Theorem 4.4.2 and Corollary 4.4.3] because our operator is not time-homogeneous as is required in the hypotheses of the preceding results.

Theorem 1.10 (Existence, uniqueness, and strong Markov property of weak solutions to a degenerate stochastic differential equation with unbounded coefficients)**.** *Suppose that the coefficients* (σ, b) *in* (1.2) *obey the conditions in Assumption 1.7. Let* $(s,x) \in [0,\infty) \times \overline{\mathbb{H}}$*. Then,*

(1) *There is a weak solution,* $(\widehat{X}, \widehat{W})$, $(\Omega, \mathscr{F}, \mathbb{P})$, $\{\mathscr{F}_t\}_{t \geq s}$, *to the stochastic differential equation* (1.2) *such that* $\widehat{X}(s) = x$, $\mathbb{P}$*-a.s.*

(2) *The weak solution is unique in the sense of probability law, that is, if*

$$(\widehat{X}^i, \widehat{W}^i), (\Omega^i, \mathscr{F}^i, \mathbb{P}^i), (\mathscr{F}^i_t)_{t \geq s}, \quad i = 1, 2,$$

are two weak solutions to the stochastic differential equation (1.2) *started at* x *at time* s*, then the two processes* X^1 *and* X^2 *have the same law.*

(3) *The unique weak solution,* $(\widehat{X}, \widehat{W})$, $(\Omega, \mathscr{F}, \mathbb{P})$, $\{\mathscr{F}_t\}_{t \geq s}$, *has the strong Markov property.*

When the initial condition (s,x) is not clear from the context, we write $X^{s,x}$ instead of X. For brevity, when the initial condition is $(0,x)$, we sometimes write X^x instead of $X^{0,x}$ or X.

Remark 1.11 (Non-exploding solutions). In the one-dimensional case, [26, Remark 5.5.19] can be applied to show that solutions to (1.2) are non-exploding; [19, Theorem 5.3.10] may also be applied to give this conclusion, noting that the moments of order $2m$ $(m \geq 1)$ are bounded via (3.13).

1.1.3. *Mimicking one-dimensional marginal probability distributions of a degenerate Itô process with unbounded coefficients.* Let X be an $\mathbb{R}^d$-valued Itô process as in (1.3), where W is an $\mathbb{R}^r$-valued Brownian motion on a filtered probability space, $(\Omega, \mathscr{F}, \mathbb{P}, \{\mathscr{F}_t\}_{t\geq 0})$, satisfying the usual conditions [26, Definition 1.2.25], β is an $\mathbb{R}^d$-valued, adapted process, and that ξ is a $\mathbb{R}^{d\times r}$-valued, adapted process satisfying the integrability condition,

$$\mathbb{E}\left[\int_0^t (|\beta(s)| + |\xi(s)\xi^*(s)|)\, ds\right] < \infty, \quad \forall t \geq 0. \tag{1.13}$$

Assume $x \in \overline{\mathbb{H}}$ and that for all $t \geq 0$ we have

$$X(t) \in \overline{\mathbb{H}}, \quad \mathbb{P}\text{-a.s.} \tag{1.14}$$

By [11, Corollary 4.5], there are (Borel) $\mathscr{B}([0,\infty)\times\overline{\mathbb{H}})$-measurable (deterministic) functions,

$$\begin{aligned} b &: [0,\infty)\times\overline{\mathbb{H}} \to \mathbb{R}^d, \\ \bar{a} &: [0,\infty)\times\overline{\mathbb{H}} \to \mathbb{S}_d, \end{aligned} \tag{1.15}$$

such that, for Lebesgue a.e. $t \geq 0$,

$$\begin{aligned} b(t, X(t)) &= \mathbb{E}\left[\beta(t)|X(t)\right] \quad \mathbb{P}\text{-a.s.}, \\ \bar{a}(t, X(t)) &= \mathbb{E}\left[\xi(t)\xi^*(t)|X(t)\right] \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{1.16}$$

We can now state the main result of this article.

Theorem 1.12 (Mimicking theorem for degenerate Itô processes with unbounded coefficients)**.** *Suppose the coefficient $\bar{a}$ in* (1.16) *satisfies* (1.10) *and the pair (a,b) obeys Assumption 2.2, where b is given by* (1.16)*. Let $\sigma \in C_{\text{loc}}([0,\infty)\times\overline{\mathbb{H}};\mathbb{R}^{d\times d})$ be a choice of square root,*

$$\bar{a}(t,x) = \sigma(t,x)\sigma^*(t,x), \quad \forall (t,x) \in [0,\infty)\times\overline{\mathbb{H}}. \tag{1.17}$$

Let $\widehat{X}$ be the unique, strong Markov weak solution to the stochastic differential equation (1.2) *started at x when $t = 0$. Then X and $\widehat{X}$ have the same one-dimensional marginal probability distributions.*

Remark 1.13 (Mimicking stochastic differential equation)*.* We call (1.2) the *mimicking stochastic differential equation* defined by the Itô process (1.3) when its coefficients are defined as in (1.9) and (1.15).

Remark 1.14 (Sufficient and necessary condition to ensure that the Itô process remains in the upper half-space)*.* In general, the coefficients σ and b_d defined by (1.16) and (1.17) are Borel measurable functions defined on $[0,\infty)\times\mathbb{R}^d$. Our assumption (1.14) implies that we may choose the coefficients σ and b_d such that they satisfy conditions (3.1) and (3.2) on $[0,\infty)\times\mathbb{R}^{d-1}\times(-\infty,0)$. Conversely, if we are given b_d and σ satisfying conditions (3.1) and (3.2), Proposition 3.8 shows that (1.14) holds.

1.2. **Connections with previous research on martingale and mimicking problems.** We briefly survey previous work on uniqueness of solutions to the martingale problem for degenerate differential operators, uniqueness and the strong Markov property for solutions to degenerate stochastic differential equations, and mimicking problems.

1.2.1. *Mimicking theorems.* Gyöngy [23, Theorem 4.6] proves existence of a mimicking process as in Theorem 1.12 — although not the uniqueness or strong Markov properties — with conditions on the coefficients (σ, b) which are both partly *weaker* than those of Theorem 1.12, because the functions $b : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are only required to be Borel-measurable, but also partly *stronger* than those of Theorem 1.12, because the functions (σ, b) are required to be uniformly bounded on $[0,\infty) \times \mathbb{R}^d$ and $\sigma\sigma^*$ is required to be uniformly positive definite on $[0,\infty) \times \mathbb{R}^d$. Since Gyöngy only requires that the coefficients (σ, b) of the corresponding mimicking stochastic differential equation (1.2) are Borel measurable functions, he uses an auxiliary regularizing procedure to construct a weak solution $widehatX$ to (1.2). Uniqueness of the weak solution is not proved under the hypotheses of [23, Theorem 4.6] and the main obstacle here is the lack of regularity of the coefficients $(\bar{a}, b)$.

The hypotheses of [23, Theorem 4.6] are quite restrictive, as we can see that they would exclude a process, X, such as that in Example 1.9, even though the coefficients of its mimicking processes, $\widehat{X}$, can be found by explicit calculation [4] (see also [2]). Moreover, Nadirashvili shows [32] that uniqueness of stochastic differential equations with *measurable* coefficients satisfying the assumptions of non-degeneracy and boundedness in [23, Theorem 4.6] does not hold in general when $d \geq 3$.

Brunick and Shreve [11, Corollary 2.16], [13] prove an extension of [23, Theorem 4.6] which relaxes the requirements that $\sigma\sigma^*$ is uniformly positive definite on $[0,\infty) \times \mathbb{R}^d$ and that the functions σ and b are bounded on $[0,\infty) \times \mathbb{R}^d$. Moreover, they significantly extend Gyöngy's theorem [23] by replacing the non-degeneracy and boundedness conditions on the coefficients of the Itô process, X, by a mild integrability condition (1.13). Using purely probabilistic methods, they show existence of weak solutions to stochastic differential equations of diffusion type which preserve not only the one-dimensional marginal distributions of the Itô process, but also certain statistics, such as the running maximum or average of one of the components. More recently, Brunick [12] establishes weak uniqueness for a degenerate stochastic differential equation with applications to pricing Asian options.

Bentata and Cont [9] and Shi and Wang [37, 39] extend Gyöngy's mimicking theorem to *discontinuous*, non-degenerate semimartingales. Under assumptions of continuity and boundedness on the coefficients of the process and non-degeneracy condition of the diffusion matrix or of the Levý operator, they prove uniqueness of solutions to the forward Kolmogorov equation associated with the generator of the mimicking process. In this setting, they show that weak uniqueness to the mimicking stochastic differential equation holds and that the mimicking process satisfies the Markov property.

Atlan [4] obtains closed-form solutions for the mimicking coefficients when the Itô process X has the form in Example 1.9, where the volatility modeled by a Bessel or Cox-Ingersoll-Ross (that is, a Feller square root) process. This is possible because explicit, tractable expressions are known for the distribution of Bessel processes and the Cox-Ingersoll-Ross process can be obtained from a Bessel process by a suitable transformation (roughly speaking, the Cox-Ingersoll-Ross process is a deterministic, time-changed Bessel process).

1.2.2. *Uniqueness and strong Markov property of solutions to degenerate stochastic differential equations.* There are counterexamples to uniqueness of weak solutions of degenerate stochastic differential equations such as (1.2); see [14], [26, §5.5], and [36]. Moreover, as noted in [11, Example 2.2.10], the Markov property of solutions to (1.2) is not guaranteed; see also [14, Example 3.10] for another example of non-Markov process arising as the solution to a one-dimensional stochastic differential equation. Sufficient conditions for weak solutions of degenerate stochastic differential

equations such as (1.2) to be Markov are provided by [33, Theorem 7.1.2], the combination [26, Theorems 5.2.9 & 5.4.20], [28] and elsewhere.

1.2.3. *Uniqueness of solutions to degenerate martingale problems.* Uniqueness for solutions to the classical martingale problem ([38, p. 138], [26, Definition 5.4.5 & 5.4.10]), for suitable coefficients (a, b), is proved in [38, Chapter 7] (see [38, §7.0] for a comprehensive outline), via uniqueness of solutions to a certain Kolmogorov backward equation. Special cases of uniqueness for solutions to the martingale problem are established in [38, Theorem 6.3.4] (via uniqueness of weak solutions to a stochastic differential equation in [38, Theorem 5.3.2]), [38, Corollary 6.3.3] (via existence and uniqueness of solutions to a parabolic partial differential equation in [38, Theorem 3.2.6]); as Stroock and Varadhan observe [38, §6.3], their special cases do not cover situations where $\sigma\sigma^*$ is only non-negative definite. Their general uniqueness result [38, §7.0] does *not* apply to the differential operator in Example 1.8 or other differential operators with similar degeneracies. Similarly, while uniqueness results for solutions to the martingale problem for certain degenerate elliptic differential operators is described by Ethier and Kurtz in [19, Theorem 8.2.5], their results do *not* apply to the differential operator in Example 1.8 or other differential operators with similar degeneracies. Well-posedness of the martingale problem for certain time-homogeneous, degenerate operators was established in [3], [7], [8] and [6], but their results do not apply to our operator (1.1) under the hypotheses of our Theorem 1.3.

The following example of Stroock and Varadhan shows that solutions to degenerate martingale problems can easily fail to be unique.

Example 1.15 (Non-uniqueness of solutions to certain degenerate martingale problems)**.** [38, Exercise 6.7.7] Consider the one-dimensional generator, $\mathscr{A}u(x) = (|x|^\alpha \wedge 1)u''(x)$, for $u \in C^2(\mathbb{R})$, with $0 < \alpha < 1$. The operator $\mathscr{A}$ is degenerate at $x = 0$ and uniqueness in law for solutions to the martingale problem for $\mathscr{A}$ fails. See [18] and [26, §5.5] for additional details. □

Additional examples of non-uniqueness of solutions to the (sub-)martingale problem are provided by Bass and Lavrentiev [6], along with suitable boundary conditions designed to achieve uniqueness.

The well-known *Yamada criterion* [40, p. 115] can be used to provide existence and uniqueness of strong solutions to one-dimensional stochastic differential equations with non-Lipschitz coefficients [40, p. 117]. A simple generalization was noted long ago by Ikeda and Watanabe [25, Theorem IV.3.2 and footnote 1, p. 182], for coefficients $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times r}$ and $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ of a stochastic differential equation such as (1.2) with $d > 1$ and $r = 1$. However, the case $d > 1$ and $r > 1$ is considerably more difficult. While there are more substantial generalizations of Yamada's theorem due to Luo [31], Altay and Schmock [1], and references cited therein and though Bogachev, Röckner, Krylov, and Zhang [10, 35] also provide related uniqueness results, they do not cover the situation to which our Theorem 1.10 applies.

1.3. **Future research.** It would be useful to establish sufficient conditions on the coefficients of the Itô process, X, which would ensure that our Assumption 2.2 on the mimicking coefficients is satisfied, or relax these assumptions further.

Because the coefficients of the stochastic differential equation (1.2) are Hölder continuous, it is natural to ask whether pathwise uniqueness for the weak solutions to the mimicking stochastic differential equation holds and so conclude that the weak solutions are actually strong. Positive answers to this question for certain degenerate stochastic differential equations are obtained by Bass, Burdzy and Chen [5].

Given an arbitrary Itô process with coefficients (ξ, β), it is difficult to determine, in general, whether the coefficients (σ, b) of the mimicking stochastic differential equation possess any further regularity than measurability. Bentata and Cont [9] assume that the coefficients of the mimicking process are continuous and they provide a sufficient condition under which this assumption is satisfied. Specifically, if the Itô process is a one-dimensional process given in the form

$$X(t) = f(Z(t)), \quad \forall t \geq 0,$$

where $f : \mathbb{R}^d \to \mathbb{R}$ is a C^2 function with bounded derivatives, $f_{x_d} \neq 0$, and Z is a "nice", $\mathbb{R}^d$-valued Markov process, then the mimicking coefficients are continuous functions. This construction is useful when one wants to reduce the dimensionality of a Markov process. In our future work, we hope to relax the local Hölder regularity condition on the mimicking coefficients and the conditions under which one can still recover the weak uniqueness of solutions.

We also plan to explore to what extent our techniques can be used in the presence of other types of degeneracy occurring in the diffusion matrix, for example, $\bar{a}_{ij}(t,x) = x_d^\alpha a_{ij}(t,x)$, with $\alpha \neq 1$, where $a(t,x)$ is uniformly elliptic. By Example 1.15, we expect that weak uniqueness will not hold for arbitrary values of α, and then one may consider the question of Markovian selection of a weak solution that mimics the one-dimensional marginal distributions of the Itô process.

1.4. **Outline of the article.** In §2, we define the Hölder spaces required to prove Theorem 2.5 (existence and uniqueness of solutions to a degenerate-parabolic partial differential equation on a half-space with unbounded coefficients) and provide a detailed description of the conditions required of the coefficients (a, b, c) in the statement of Theorem 2.5. Section 3 contains the proofs of Theorems 1.3, 1.10, and 1.12. In §3.1, we prove existence of solutions to the degenerate martingale problem and degenerate stochastic differential equation specified in Theorems 1.3 and 1.10, while in §3.2, we prove uniqueness and the strong Markov property in Theorems 1.3 and 1.10. Lastly, in §3.3, we prove our mimicking theorem for a degenerate Itô process, namely, Theorem 1.12.

1.5. **Notation and conventions.** We adopt the convention that a condition labeled as an *Assumption* is considered to be universal and in effect throughout this article and so not referenced explicitly in theorem and similar statements; a condition labeled as a *Hypothesis* is only considered to be in effect when explicitly referenced.

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2. Weighted Hölder spaces and coefficients of the differential operators

In §2.1, we introduce the Hölder spaces required for the statement and proof of Theorem 2.5, while in §2.2, we describe the regularity and growth conditions required of the coefficients (a, b, c) in Theorem 2.5. In 2.3, we recall our main result, Theorem 2.5, from [21].

2.1. **Weighted Hölder spaces.** For $a > 0$, we denote

$$\mathbb{H}_{a,T} := (0,T) \times \mathbb{R}^{d-1} \times (0,a),$$

and, when $T = \infty$, we denote $\mathbb{H}_\infty = (0,\infty) \times \mathbb{H}$ and $\mathbb{H}_{a,\infty} = (0,\infty) \times \mathbb{R}^{d-1} \times (0,a)$. We denote the usual closures these half-spaces and cylinders by $\overline{\mathbb{H}} := \mathbb{R}^{d-1} \times [0,\infty)$, $\overline{\mathbb{H}}_T := [0,T] \times \overline{\mathbb{H}}$, while

$\overline{\mathbb{H}}_{a,T} := [0,T]\times\mathbb{R}^{d-1}\times[0,a]$. We write points in $\mathbb{H}$ as $x := (x', x_d)$, where $x' := (x_1, x_2, \dots, x_{d-1}) \in \mathbb{R}^{d-1}$. For $x^0 \in \overline{\mathbb{H}}$ and $R > 0$, we let

$$\begin{aligned} B_R(x^0) &:= \left\{x \in \mathbb{H} : |x - x^0| < R\right\}, \\ Q_{R,T}(x^0) &:= (0,T) \times B_R(x^0), \end{aligned}$$

and denote their usual closures by $\bar{B}_R(x^0) := \{x \in \mathbb{H} : |x - x^0| \leq R\}$ and $\bar{Q}_{R,T}(x^0) := [0,T] \times \bar{B}_R(x^0)$, respectively. We write B_R or $Q_{R,T}$ when the center, x^0, is clear from the context or unimportant.

A parabolic partial differential equation with a degeneracy similar to that considered in this article arises in the study of the porous medium equation [15, 16, 27]. The existence, uniqueness, and regularity theory for such equations is facilitated by the use of Hölder spaces defined by the *cycloidal metric* on $\mathbb{H}$ introduced by Daskalopoulos and Hamilton [15] and, independently, by Koch [27]. See [15, p. 901] for a discussion of this metric. Following [15, p. 901], we define the *cycloidal distance* between two points, $P_1 = (t_1, x^1), P_2 = (t_2, x^2) \in [0,\infty) \times \overline{\mathbb{H}}$, by

$$s(P_1, P_2) := \frac{\sum_{i=1}^d |x_i^1 - x_i^2|}{\sqrt{x_d^1} + \sqrt{x_d^2} + \sqrt{\sum_{i=1}^{d-1} |x_i^1 - x_i^2|}} + \sqrt{|t_1 - t_2|}. \tag{2.1}$$

Following [29, p. 117], we define the usual Euclidean distance between points $P_1, P_2 \in [0,\infty)\times\mathbb{R}^d$ by

$$\rho(P_1, P_2) := \sum_{i=1}^d |x_i^1 - x_i^2| + \sqrt{|t_1 - t_2|}. \tag{2.2}$$

Remark 2.1 (Equivalence of the cycloidal and Euclidean distance functions on suitable subsets of $[0,\infty) \times \mathbb{H}$). The cycloidal and Euclidean distance functions, s and ρ, are equivalent on sets of the form $[0,\infty) \times \mathbb{R}^{d-1} \times [y_0, y_1]$, for any $0 < y_0 < y_1$.

Let $\Omega \subset (0,T) \times \mathbb{H}$ be an open set and $\alpha \in (0,1)$. We denote by $C(\bar{\Omega})$ the space of bounded, continuous functions on $\bar{\Omega}$, and by $C_0^\infty(\bar{\Omega})$ the space of smooth functions with compact support in $\bar{\Omega}$. For a function $u : \bar{\Omega} \to \mathbb{R}$, we consider the following norms and seminorms

$$\|u\|_{C(\bar{\Omega})} = \sup_{P\in\bar{\Omega}} |u(P)|, \tag{2.3}$$

$$[u]_{C_s^\alpha(\bar{\Omega})} = \sup_{\substack{P_1,P_2\in\bar{\Omega},\\ P_1\neq P_2}} \frac{|u(P_1) - u(P_2)|}{s^\alpha(P_1,P_2)}, \tag{2.4}$$

$$[u]_{C_\rho^\alpha(\bar{\Omega})} = \sup_{\substack{P_1,P_2\in\bar{\Omega},\\ P_1\neq P_2}} \frac{|u(P_1) - u(P_2)|}{\rho^\alpha(P_1,P_2)}. \tag{2.5}$$

We say that $u \in C_s^\alpha(\bar{\Omega})$ if $u \in C(\bar{\Omega})$ and

$$\|u\|_{C_s^\alpha(\bar{\Omega})} = \|u\|_{C(\bar{\Omega})} + [u]_{C_s^\alpha(\bar{\Omega})} < \infty.$$

Analogously, we define the Hölder space $C_\rho^\alpha(\bar{\Omega})$ of functions u which satisfy

$$\|u\|_{C_\rho^\alpha(\bar{\Omega})} = \|u\|_{C(\bar{\Omega})} + [u]_{C_\rho^\alpha(\bar{\Omega})} < \infty.$$

We say that $u \in C_s^{2+\alpha}(\bar{\Omega})$ if

$$\|u\|_{C_s^{2+\alpha}(\bar{\Omega})} := \|u\|_{C_s^\alpha(\bar{\Omega})} + \|u_t\|_{C_s^\alpha(\bar{\Omega})} + \max_{1\leq i\leq d} \|u_{x_i}\|_{C_s^\alpha(\bar{\Omega})} + \max_{1\leq i,j\leq d} \|x_d u_{x_ix_j}\|_{C_s^\alpha(\bar{\Omega})} < \infty,$$

and $u \in C^{2+\alpha}_\rho(\bar\Omega)$ if

$$\|u\|_{C^{2+\alpha}_\rho(\bar\Omega)} = \|u\|_{C^\alpha_\rho(\bar\Omega)} + \|u_t\|_{C^\alpha_\rho(\bar\Omega)} + \max_{1\le i\le d} \|u_{x_i}\|_{C^\alpha_\rho(\bar\Omega)} + \max_{1\le i,j\le d} \|u_{x_ix_j}\|_{C^\alpha_\rho(\bar\Omega)} < \infty.$$

We denote by $C^\alpha_{s,\text{loc}}(\bar\Omega)$ the space of functions u with the property that for any compact set $K \subseteq \bar\Omega$, we have $u \in C^\alpha_s(K)$. Analogously, we define the spaces $C^{2+\alpha}_{s,\text{loc}}(\bar\Omega)$, $C^\alpha_{\rho,\text{loc}}(\bar\Omega)$ and $C^{2+\alpha}_{\rho,\text{loc}}(\bar\Omega)$.

We prove existence, uniqueness and regularity of solutions for a parabolic operator (2.11) whose second order coefficients are degenerate on $\partial\mathbb{H}$. For this purpose, we will make use of the following Hölder spaces

$$\begin{aligned}
\mathscr{C}^\alpha(\overline{\mathbb{H}}_T) &:= \left\{u : u \in C^\alpha_s(\overline{\mathbb{H}}_{1,T}) \cap C^\alpha_\rho(\overline{\mathbb{H}}_T \setminus \mathbb{H}_{1,T})\right\},\\
\mathscr{C}^{2+\alpha}(\overline{\mathbb{H}}_T) &:= \left\{u : u \in C^{2+\alpha}_s(\overline{\mathbb{H}}_{1,T}) \cap C^{2+\alpha}_\rho(\overline{\mathbb{H}}_T \setminus \mathbb{H}_{1,T})\right\}.
\end{aligned}$$

We define $\mathscr{C}^\alpha(\overline{\mathbb{H}})$ and $\mathscr{C}^{2+\alpha}(\overline{\mathbb{H}})$ in the analogous manner.

The coefficient functions $x_d a_{ij}(t,x)$, $b_i(t,x)$ and $c(t,x)$ of the parabolic operator (2.11) are allowed to have linear growth in $|x|$. To account for the unboundedness of the coefficients, we augment our definition of Hölder spaces by introducing weights of the form $(1+|x|)^q$, where $q \ge 0$ will be suitably chosen in the sequel. For $q \ge 0$, we define

$$\|u\|_{\mathscr{C}^0_q(\overline{\mathbb{H}})} := \sup_{x\in\overline{\mathbb{H}}} (1+|x|)^q\,|u(x)|, \tag{2.6}$$

and, given $T > 0$, we define

$$\|u\|_{\mathscr{C}^0_q(\overline{\mathbb{H}}_T)} := \sup_{(t,x)\in\overline{\mathbb{H}}_T} (1+|x|)^q\,|u(t,x)|. \tag{2.7}$$

Moreover, given $\alpha \in (0,1)$, we define

$$\|u\|_{\mathscr{C}^\alpha_q(\overline{\mathbb{H}}_T)} := \|u\|_{\mathscr{C}^0_q(\overline{\mathbb{H}}_T)} + [(1+|x|)^q u]_{C^\alpha_s(\overline{\mathbb{H}}_{1,T})} + [(1+|x|)^q u]_{C^\alpha_\rho(\overline{\mathbb{H}}_T\setminus\mathbb{H}_{1,T})}, \tag{2.8}$$

$$\|u\|_{\mathscr{C}^{2+\alpha}_q(\overline{\mathbb{H}}_T)} := \|u\|_{\mathscr{C}^\alpha_q(\overline{\mathbb{H}}_T)} + \|u_t\|_{\mathscr{C}^\alpha_q(\overline{\mathbb{H}}_T)} + \|u_{x_i}\|_{\mathscr{C}^\alpha_q(\overline{\mathbb{H}}_T)} + \|x_d u_{x_ix_j}\|_{\mathscr{C}^\alpha_q(\overline{\mathbb{H}}_T)}. \tag{2.9}$$

The vector spaces

$$\begin{aligned}
\mathscr{C}^0_q(\overline{\mathbb{H}}_T) &:= \left\{u \in C(\overline{\mathbb{H}}_T) : \|u\|_{\mathscr{C}^0_q(\overline{\mathbb{H}}_T)} < \infty\right\},\\
\mathscr{C}^\alpha_q(\overline{\mathbb{H}}_T) &:= \left\{u \in \mathscr{C}^\alpha(\overline{\mathbb{H}}_T) : \|u\|_{\mathscr{C}^\alpha_q(\overline{\mathbb{H}}_T)} < \infty\right\},\\
\mathscr{C}^{2+\alpha}_q(\overline{\mathbb{H}}_T) &:= \left\{u \in \mathscr{C}^{2+\alpha}(\overline{\mathbb{H}}_T) : \|u\|_{\mathscr{C}^{2+\alpha}_q(\overline{\mathbb{H}}_T)} < \infty\right\},
\end{aligned}$$

can be shown to be Banach spaces with respect to the norms (2.7), (2.8) and (2.9), respectively. We define the vector spaces $\mathscr{C}^0_q(\overline{\mathbb{H}})$, $\mathscr{C}^\alpha_q(\overline{\mathbb{H}})$, and $\mathscr{C}^{2+\alpha}_q(\overline{\mathbb{H}})$ similarly, and each can be shown to be a Banach space when equipped with the corresponding norm.

We let $\mathscr{C}^{2+\alpha}_{q,\text{loc}}(\overline{\mathbb{H}}_T)$ denote the vector space of functions u such that for any compact set $K \subset \overline{\mathbb{H}}_T$, we have $u \in \mathscr{C}^{2+\alpha}_{q,\text{loc}}(K)$, for all $q \ge 0$.

When $q = 0$, the subscript q is omitted in the preceding definitions.

2.2. **Coefficients of the differential operators.** In our companion article [21], we used a Schauder approach to prove existence, uniqueness, and regularity of solutions to the degenerate-parabolic partial differential equation,

$$\begin{cases} Lu = f & \text{on } \mathbb{H}_T,\\ u(0,\cdot) = g & \text{on } \overline{\mathbb{H}}, \end{cases} \tag{2.10}$$

where

$$-Lu = -u_t + \sum_{i,j=1}^{d} x_d a_{ij} u_{x_i x_j} + \sum_{i=1}^{d} b_i u_{x_i} + cu, \quad \forall\, u \in C^{1,2}(\mathbb{H}_T). \tag{2.11}$$

Observe that

$$-Lu = -u_t + \mathscr{A}_t u + cu,$$

where $\mathscr{A}_t$ is given in (1.1), provided we absorb the factor $1/2$ into the definition of the coefficients, a_{ij}. Unless other conditions are explicitly substituted, we require in this article that the coefficients (a, b, c) of the parabolic differential operator L in (2.11) satisfy the conditions in the following

Assumption 2.2 (Properties of the coefficients of the parabolic differential operator)**.** There are constants $\delta > 0$, $K > 0$, $\nu > 0$ and $\alpha \in (0, 1)$ such that the following hold.

(1) The coefficients c and b_d obey

$$c(t, x) \leq K, \quad \forall (t, x) \in \overline{\mathbb{H}}_\infty, \tag{2.12}$$

$$b_d(t, x', 0) \geq \nu, \quad \forall (t, x') \in [0, \infty) \times \mathbb{R}^{d-1}. \tag{2.13}$$

(2) On $\overline{\mathbb{H}}_{2,\infty}$ (that is, near $x_d = 0$), we require that

$$\sum_{i,j=1}^{d} a_{ij}(t, x)\eta_i\eta_j \geq \delta|\eta|^2, \quad \forall \eta \in \mathbb{R}^d, \quad \forall (t, x) \in \overline{\mathbb{H}}_{2,\infty}, \tag{2.14}$$

$$\max_{1\leq i,j\leq d} \|a_{ij}\|_{C(\overline{\mathbb{H}}_{2,\infty})} + \max_{1\leq i\leq d} \|b_i\|_{C(\overline{\mathbb{H}}_{2,\infty})} + \|c\|_{C(\overline{\mathbb{H}}_{2,\infty})} \leq K, \tag{2.15}$$

and, for all $P_1, P_2 \in \overline{\mathbb{H}}_{2,\infty}$ such that $P_1 \neq P_2$ and $s(P_1, P_2) \leq 1$,

$$\begin{aligned}
\max_{1\leq i,j\leq d} \frac{|a_{ij}(P_1) - a_{ij}(P_2)|}{s^\alpha(P_1, P_2)} &\leq K, \\
\max_{1\leq i\leq d} \frac{|b_i(P_1) - b_i(P_2)|}{s^\alpha(P_1, P_2)} &\leq K, \\
\frac{|c(P_1) - c(P_2)|}{s^\alpha(P_1, P_2)} &\leq K.
\end{aligned} \tag{2.16}$$

(3) On $\overline{\mathbb{H}}_\infty \setminus \mathbb{H}_{2,\infty}$ (that is, farther away from $x_d = 0$), we require that

$$\sum_{i,j=1}^{d} x_d a_{ij}(t, x)\eta_i\eta_j \geq \delta|\eta|^2, \quad \forall \eta \in \mathbb{R}^d, \quad \forall (t, x) \in \overline{\mathbb{H}}_\infty \setminus \mathbb{H}_{2,\infty}, \tag{2.17}$$

and, for all $P_1, P_2 \in \overline{\mathbb{H}}_\infty \setminus \mathbb{H}_{2,\infty}$ such that $P_1 \neq P_2$ and $\rho(P_1, P_2) \leq 1$,

$$\begin{aligned}
\max_{1\leq i,j\leq d} \frac{|x_d^1 a_{ij}(P_1) - x_d^2 a_{ij}(P_2)|}{\rho^\alpha(P_1, P_2)} &\leq K, \\
\max_{1\leq i\leq d} \frac{|b_i(P_1) - b_i(P_2)|}{\rho^\alpha(P_1, P_2)} &\leq K, \\
\frac{|c(P_1) - c(P_2)|}{\rho^\alpha(P_1, P_2)} &\leq K.
\end{aligned} \tag{2.18}$$

Remark 2.3 (Local Hölder conditions on the coefficients). The local Hölder conditions (2.16) and (2.18) are similar to those in [30, Hypothesis 2.1].

Remark 2.4 (Linear growth of the coefficients of the parabolic differential operator). Conditions (2.15) and (2.18) imply that the coefficients $x_d a_{ij}(t,x)$, $b_i(t,x)$ and $c(t,x)$ can have at most linear growth in x. In particular, we may choose the constant K large enough such that

$$\sum_{i,j=1}^{d} |x_d a_{ij}(t,x)| + \sum_{i=1}^{d} |b_i(t,x)| + |c(t,x)| \leq K(1+|x|), \quad \forall (t,x) \in \overline{\mathbb{H}}_\infty. \tag{2.19}$$

2.3. **Existence and uniqueness of solutions to a degenerate-parabolic partial differential equation with unbounded coefficients.** We recall our main result from [21].

Theorem 2.5 (Existence and uniqueness of solutions to a degenerate-parabolic partial differential equation with unbounded coefficients)**.** [21, Theorem 1.1] *Assume that the coefficients* (a,b,c) *in* (2.11) *obey the conditions in Assumption 2.2. Then there is a positive constant* p*, depending only on the Hölder exponent* $\alpha \in (0,1)$*, such that for any* $T>0$*,* $f \in \mathscr{C}^{\alpha}_p(\overline{\mathbb{H}}_T)$ *and* $g \in \mathscr{C}^{2+\alpha}_p(\overline{\mathbb{H}})$*, there exists a unique solution* $u \in \mathscr{C}^{2+\alpha}(\overline{\mathbb{H}}_T)$ *to* (2.10)*. Moreover,* u *satisfies the a priori estimate*

$$\|u\|_{\mathscr{C}^{2+\alpha}(\overline{\mathbb{H}}_T)} \leq C \left(\|f\|_{\mathscr{C}^{\alpha}_p(\overline{\mathbb{H}}_T)} + \|g\|_{\mathscr{C}^{2+\alpha}_p(\overline{\mathbb{H}})} \right), \tag{2.20}$$

where C *is a positive constant, depending only on* K*,* ν*,* δ*,* d*,* α *and* T*.*

3. Martingale problem and the mimicking theorem

In this section, we prove Theorem 1.10 concerning the degenerate stochastic differential equation with unbounded coefficients (1.2), and establish the main result, Theorem 1.12. Our method of proof combines ideas from the martingale problem formulation of Stroock and Varadhan [38] and the existence of solutions in suitable Hölder spaces, $\mathscr{C}^{2+\alpha}(\overline{\mathbb{H}}_T)$, to the homogeneous version of the initial value problem established in Theorem 2.5. In §3.1, we prove existence of weak solutions to the mimicking stochastic differential equation (1.2) and the existence of solutions to the martingale problem associated to $\mathscr{A}_t$. In §3.2, we establish uniqueness in law of solutions to (1.2) and to the martingale problem for $\mathscr{A}_t$, thus proving Theorems 1.10 and 1.3; in §3.3, we establish the matching property for the one-dimensional probability distributions for solutions to (1.2) and of an Itô process, thus proving Theorem 1.12.

3.1. **Existence of solutions to the martingale problem and of weak solutions to the stochastic differential equation.** In this subsection, we show that (1.2) has weak solutions $(\widehat{X}, \widehat{W})$ on some filtered probability space $(\Omega, \mathscr{F}, \mathbb{P})$, $\{\mathscr{F}_t\}_{t\geq 0}$ [26, Definition 5.3.1], for any initial point $x \in \overline{\mathbb{H}}$, by proving existence of solutions to the martingale problem associated to $\mathscr{A}_t$ (Definition 1.1).

We begin with an intuitive property of solutions to (1.2) defined by an initial condition in $\overline{\mathbb{H}}$. For this purpose, we consider coefficients defined on $[0,\infty) \times \mathbb{R}^d$, instead of $[0,\infty) \times \overline{\mathbb{H}}$. While Proposition 3.1 could also be proved using the Tanaka formula, the simple proof we give below avoids the need to consider the local time of the process $\widehat{X}_d(t)$.

Proposition 3.1 (Solutions started in a half-space remain in a half-space)**.** *Let*

$$\tilde{\sigma} : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^{d\times d},$$
$$\tilde{b} : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d,$$

be Borel measurable functions. Assume that

$$\tilde{\sigma}(t,x) = 0 \quad \textit{when } x_d < 0, \tag{3.1}$$

and b satisfies

$$0 \leq \tilde{b}_d(t,x) \leq K \quad \text{when } x_d < 0, \tag{3.2}$$

where K is a positive constant. Let $\widehat{X}$ be a weak solution of

$$d\widehat{X} = \tilde{b}(t,\widehat{X}(t))dt + \tilde{\sigma}(t,\widehat{X}(t))d\widehat{W}(t), \quad t \geq s,$$

such that $\widehat{X}(s) \in \overline{\mathbb{H}}$. Then

$$\mathbb{P}\left(\widehat{X}(t) \in \overline{\mathbb{H}}\right) = 1, \quad \forall t \geq s. \tag{3.3}$$

Proof. It is sufficient to show that for any $\varepsilon > 0$, we have

$$\mathbb{P}\left(\widehat{X}_d(t) \in (-\infty, -\varepsilon)\right) = 0, \quad \forall t \geq s. \tag{3.4}$$

Let $\varphi : \mathbb{R} \to [0,1]$ be a smooth, non-negative cutoff function such that

$$\varphi|_{(-\infty,-\varepsilon)} \equiv 1, \ \varphi|_{(0,\infty)} \equiv 0, \text{ and } \varphi' \leq 0. \tag{3.5}$$

Then, by the Itô lemma [26, Theorem 3.3.3], we obtain

$$\begin{aligned}\varphi(\widehat{X}_d(t)) &= \varphi(\widehat{X}_d(s)) + \int_s^t \sum_{i=0}^d \tilde{\sigma}_{di}(v,\widehat{X}(v))\varphi'(\widehat{X}_d(v))\, d\widehat{W}_i(v) \\ &\quad + \int_s^t \left[\tilde{b}_d(v,\widehat{X}(v))\varphi'(\widehat{X}_d(v)) + \frac{1}{2}(\tilde{\sigma}\tilde{\sigma}^*)_{dd}(v,\widehat{X}(v))\varphi''(\widehat{X}_d(v))\right] dv,\end{aligned}$$

and so, because $\operatorname{supp}\varphi \subset (-\infty, 0]$ and (3.1) is satisfied, we have

$$\varphi(\widehat{X}_d(t)) = \varphi(\widehat{X}_d(s)) + \int_s^t \tilde{b}_d(v,\widehat{X}(v))\varphi'(\widehat{X}_d(v))\, dv.$$

By (3.2) and (3.5), the integral term in the preceding identity is non-positive. Therefore, we must have $\varphi(\widehat{X}_d(t)) \leq 0$ and hence $\varphi(\widehat{X}_d(t)) = 0$, for any choice of $\varepsilon > 0$, from where (3.4) and then (3.3) follow. □

Remark 3.2 (Weak solutions are independent of choice of extension of coefficients to lower half-space). Let

$$\tilde{b}^i : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d, \quad i = 1, 2,$$

be measurable functions which satisfy condition (3.2), and assume

$$\tilde{b}^1 = \tilde{b}^2 \quad \text{on} \quad [0,\infty) \times \overline{\mathbb{H}}. \tag{3.6}$$

Let $\tilde{\sigma}$ be a measurable function as in the hypotheses of Proposition 3.1. Let $\widehat{X}$ be a weak solution to

$$d\widehat{X}(t) = \tilde{b}^1(t,\widehat{X}(t))dt + \tilde{\sigma}(t,\widehat{X}(t))d\widehat{W}(t), \quad \forall t \geq s, \tag{3.7}$$

such that

$$\mathbb{P}\left(\widehat{X}(s) \in \overline{\mathbb{H}}\right) = 1.$$

Then, Proposition 3.1 shows that $\widehat{X}(t)$ remains supported in $\overline{\mathbb{H}}$, for all $t \geq s$. By (3.6), it follows that $\widehat{X}$ is a weak solution to

$$d\widehat{X}(t) = \tilde{b}^2(t,\widehat{X}(t))dt + \tilde{\sigma}(t,\widehat{X}(t))d\widehat{W}(t), \quad \forall t \geq s. \tag{3.8}$$

This simple observation shows that, under the hypotheses of Proposition 3.1, any weak solution started in $\overline{\mathbb{H}}$ to (3.7) is a weak solution to (3.8), and vice versa.

We have the following consequence of the existence theorem [19, Theorem 5.3.10] of weak solutions for stochastic differential equations with unbounded, continuous coefficients. The main difference between [19, Theorem 5.3.10] and Theorem 3.3 is that the coefficients, b and σ, in (1.2) are defined on $[0,\infty)\times\bar{\mathbb{H}}$, while the coefficients of the stochastic differential equation considered in [19, Theorem 5.3.10] are defined on $[0,\infty)\times\mathbb{R}^d$.

Theorem 3.3 (Existence of weak solutions to a stochastic differential equation with continuous coefficients)**.** *Assume that the coefficients σ and b in* (1.2) *are continuous on $[0,\infty)\times\overline{\mathbb{H}}$, that $\bar{a}$ obeys condition* (1.10) *on $[0,\infty)\times\overline{\mathbb{H}}$, and that $\bar{a}$ and b have at most linear growth in the spatial variable, that is, condition* (2.19) *holds. Then,*

1. *For any $(s,x)\in[0,\infty)\times\overline{\mathbb{H}}$, there exist weak solutions $(\widehat{X},\widehat{W})$, $(\Omega,\mathscr{F},\mathbb{P})$, $(\mathscr{F}_t)_{t\geq s}$, to* (1.2) *such that $\widehat{X}(s)=x$.*
2. *For any $(s,x)\in[0,\infty)\times\overline{\mathbb{H}}$, there is a solution, $\widehat{\mathbb{P}}^{s,x}$, to the martingale problem associated to $\mathscr{A}_t$ such that* (1.5) *holds.*

Proof. Because $\bar{a}\in C_{\text{loc}}([0,\infty)\times\overline{\mathbb{H}})$ satisfies condition (1.10), we may extend σ as a continuous function to $[0,\infty)\times\mathbb{R}^d$ such that (3.1) is satisfied. We denote this extension by $\tilde{\sigma}\in C_{\text{loc}}([0,\infty)\times\mathbb{R}^d)$. Similarly, we consider an extension, $\tilde{b}\in C_{\text{loc}}([0,\infty)\times\mathbb{R}^d)$, of the coefficient b in (1.16), such that (3.2) is satisfied. Let $(\widehat{X},\widehat{W})$ be a weak solution, given by [19, Theorem 5.3.10], to the stochastic differential equation

$$d\widehat{X}=\tilde{b}(t,\widehat{X}(t))dt+\tilde{\sigma}(t,\widehat{X}(t))d\widehat{W}(t),\quad t>s,\quad \widehat{X}(s)=x.$$

Because we assume $\widehat{X}(s)=x\in\bar{\mathbb{H}}$, Proposition 3.1 shows that $\mathbb{P}(\widehat{X}(t)\in\bar{\mathbb{H}})=1$, for all $t\geq s$, and so, we see that $(\widehat{X},\widehat{W})$ is also a weak solution to (1.2). This proves the first statement of the theorem. To prove the second statement of the theorem, let $\widehat{\mathbb{P}}^{s,x}$ be the probability measure induced by the weak solution $\widehat{X}$ on $(C_{\text{loc}}([0,\infty);\overline{\mathbb{H}}),\mathscr{B}(C_{\text{loc}}([0,\infty);\overline{\mathbb{H}})))$. Then, [26, Problem 5.4.3] implies that $\widehat{\mathbb{P}}^{s,x}$ is a solution to the martingale problem associated to $\mathscr{A}_t$ and satisfies (1.5). □

3.2. **Uniqueness of solutions to the martingale problem and of weak solutions to the stochastic differential equation.** We show that uniqueness in the sense of probability law holds for the weak solutions of the stochastic differential equation (1.2), with initial condition $x\in\overline{\mathbb{H}}$, and we establish the well-posedness of the martingale problem associated to (1.2). First, we prove that uniqueness of the one-dimensional marginal distributions holds for weak solutions to (1.2), and then the analogue of [26, Proposition 5.4.27] is used to show that uniqueness in law of solutions also holds.

We begin with the following version of the standard Itô lemma (compare [26, Theorem 3.3.6]) which applies to Itô processes which are solutions to (1.2). The standard Itô lemma cannot be applied directly because the function v is not C^2 up to the boundary (we only know that x_dD^2v is continuous up to the boundary). So we shall apply the standard Itô lemma to the processes $X^\varepsilon=X\vee\varepsilon$, let ε tend to zero, and use conditions (3.10) and (3.11) to prove the appropriate convergence.

Proposition 3.4 (Itô lemma)**.** *Assume that the coefficients σ and b of* (1.2) *are Borel measurable functions, $\bar{a}$ obeys condition* (1.10) *on $[0,\infty)\times\overline{\mathbb{H}}$, and $\bar{a}$ and b have at most linear growth in the spatial variable, that is, condition* (2.19) *holds. Assume there is a positive constant K such that*

$$|a_{ij}(t,x)|\leq K\quad\forall(t,x)\in[0,T]\times\mathbb{R}^{d-1}\times[0,1]. \tag{3.9}$$

Let $v \in C_{\mathrm{loc}}([0,\infty)\times\overline{\mathbb{H}})$ be such that it satisfies, for all $1 \le i,j \le d$,

$$v_t, v_{x_i}, x_d v_{x_ix_j} \in C_{\mathrm{loc}}([0,\infty)\times\overline{\mathbb{H}}), \tag{3.10}$$

$$x_d v_{x_ix_j} = 0 \quad \text{on } [0,T]\times\partial\mathbb{H}. \tag{3.11}$$

Let $(\widehat{X},\widehat{W})$ be a weak solution to (1.2) on a filtered probability space $(\Omega,\mathbb{P},\mathscr{F})$, $\{\mathscr{F}_t\}_{t\ge 0}$, such that $\widehat{X}(0)\in\overline{\mathbb{H}}$, $\mathbb{P}$-a.s. Then, the following holds $\mathbb{P}$-a.s., for all $0\le t\le T$,

$$\begin{aligned}
v(t,\widehat{X}(t)) = v(0,\widehat{X}(0)) &+ \int_0^t \sum_{i,j=1}^d \sigma_{ij}(u,\widehat{X}(u))v_{x_j}(u,\widehat{X}(u))\,d\widehat{W}_j(u)\\
&+ \int_0^t \left(\sum_{i=1}^d b_i(u,\widehat{X}(u))v_{x_i}(u,\widehat{X}(u))\right.\\
&\left.+\sum_{i,j=1}^d \frac{1}{2}\widehat{X}_d(u)a_{ij}(u,\widehat{X}(u))v_{x_ix_j}(u,\widehat{X}(u))\right)\,du.
\end{aligned}\tag{3.12}$$

Proof. The proof follows by applying the standard Itô lemma, [26, Theorem 3.3.6], to the processes $X^\varepsilon := X\vee\varepsilon$, for $\varepsilon>0$, and taking limit as ε tends to zero. This will require the use of conditions (3.10) and (3.11). We choose $\varepsilon>0$ and let

$$\begin{aligned}
x^\varepsilon &:= (x_1,\dots,x_{d-1},x_d+\varepsilon),\\
\widehat{X}^\varepsilon(u) &:= \left(\widehat{X}_1(u),\dots,\widehat{X}_{d-1}(u),\widehat{X}_d(u)+\varepsilon\right),\quad \forall u\ge 0.
\end{aligned}$$

Consider the stopping times

$$\tau_n := \inf\left\{u\ge 0 : |\widehat{X}(u)|\ge n\right\}\quad \forall n\ge 1.$$

Since the coefficients $\bar{a}$ and b have at most linear growth in the spatial variable (condition (2.19) holds), we obtain by [26, Problem 5.3.15], that for all $m\ge 1$ and $t\ge 0$, there is a positive constant $C=C(m,t,K,d)$ such that

$$\mathbb{E}\left[\max_{0\le u\le t}|\widehat{X}(u)|^{2m}\right] \le C\left(1+|x|^{2m}\right). \tag{3.13}$$

Then, it follows by (3.13) that the non-decreasing sequence of stopping times $\{\tau_n\}_{n\ge1}$ satisfies

$$\lim_{n\to\infty}\tau_n = +\infty \quad \mathbb{P}\text{-a.s.} \tag{3.14}$$

If this were not the case, then there is $t>0$ such that

$$\lim_{n\to\infty}\mathbb{P}(\tau_n\le t) > 0. \tag{3.15}$$

But, $\mathbb{P}(\tau_n\le t) = \mathbb{P}\left(\sup_{0\le u\le t}|\widehat{X}(u)|\ge n\right)$ and we have

$$\begin{aligned}
\mathbb{P}\left(\sup_{0\le u\le t}|\widehat{X}(u)|\ge n\right) &\le \frac{1}{n^2}\mathbb{E}\left[\max_{0\le u\le t}|\widehat{X}(u)|^2\right]\\
&\le \frac{C(1+|x|^2)}{n^2}, \quad \text{(by (3.13))}.
\end{aligned}$$

Since the preceding expression converges to zero, as n goes to ∞, we obtain a contradiction in (3.15), and so (3.14) holds. By (3.14), it suffices to prove (3.12) for the stopped process, that is

$$\begin{aligned} v(t\wedge\tau_n, \widehat{X}(t\wedge\tau_n)) &= v(0,\widehat{X}(0)) + \int_0^{t\wedge\tau_n}\sum_{i,j=1}^d \sigma_{ij}(u,\widehat{X}(u))v_{x_j}(u,\widehat{X}(u))\,d\widehat{W}_j(u)\\ &\quad+\int_0^{t\wedge\tau_n}\left(\sum_{i=1}^d b_i(u,\widehat{X}(u))v_{x_i}(u,\widehat{X}(u))\right.\\ &\quad\left.+\sum_{i,j=1}^d\frac{1}{2}\widehat{X}_d(u)a_{ij}(u,\widehat{X}(u))v_{x_ix_j}(u,\widehat{X}(u))\right)\,du. \end{aligned} \tag{3.16}$$

By Proposition 3.1, we have

$$\widehat{X}(u)\in\overline{\mathbb{H}}\quad \mathbb{P}\text{-a.s.}\quad \forall u\in[0,T]. \tag{3.17}$$

Since $v\in C^{1,2}_{\mathrm{loc}}([0,T]\times\mathbb{R}^{d-1}\times[\varepsilon/2,\infty))$, we may extend v to be a $C^{1,2}_{\mathrm{loc}}$ function on $[0,T]\times\mathbb{R}^d$. Then we can apply the standard Itô lemma, [26, Theorem 3.3.6] and, taking into account that $\widehat{X}(t)+\varepsilon\geq\varepsilon$, $\mathbb{P}$-a.s., for all $t\geq 0$, we obtain

$$\begin{aligned} v(t\wedge\tau_n, \widehat{X}^\varepsilon(t\wedge\tau_n)) &= v(0,\widehat{X}^\varepsilon(0)) + \int_0^{t\wedge\tau_n}\sum_{i,j=1}^d \sigma_{ij}(u,\widehat{X}(u))v_{x_j}(u,\widehat{X}^\varepsilon(u))\,d\widehat{W}_j(u)\\ &\quad+\int_0^{t\wedge\tau_n}\left(v_t(u,\widehat{X}^\varepsilon(u))+\sum_{i=1}^d b_i(u,\widehat{X}(u))v_{x_i}(u,\widehat{X}^\varepsilon(u))\right.\\ &\quad\left.+\sum_{i,j=1}^d\frac{1}{2}\widehat{X}_d(u)a_{ij}(u,\widehat{X}(u))v_{x_ix_j}(u,\widehat{X}^\varepsilon(u))\right)\,du. \end{aligned} \tag{3.18}$$

Our goal is to show that, by taking the limit as $\varepsilon\downarrow 0$ in the preceding equation, we obtain (3.16).

Since $v\in C_{\mathrm{loc}}(\overline{\mathbb{H}}_T)$, we have for all $0\leq u\leq T$,

$$v(u\wedge\tau_n,\widehat{X}^\varepsilon(u\wedge\tau_n))\to v(u\wedge\tau_n,\widehat{X}(u\wedge\tau_n))\quad \mathbb{P}\text{-a.s. when } \varepsilon\downarrow 0. \tag{3.19}$$

The terms in (3.18) containing the pure Itô integrals can be evaluated in the following way. As usual, we have

$$\begin{aligned} &\mathbb{E}\left[\left|\int_0^{t\wedge\tau_n}\sigma_{ij}(u,\widehat{X}(u))v_{x_j}(u,\widehat{X}^\varepsilon(u))\,d\widehat{W}_j(u)-\int_0^{t\wedge\tau_n}\sigma_{ij}(u,\widehat{X}(u))v_{x_j}(u,\widehat{X}(u))\,d\widehat{W}_j(u)\right|\right]\\ &\leq\mathbb{E}\left[\left|\int_0^{t\wedge\tau_n}\sigma_{ij}(u,\widehat{X}(u))\left(v_{x_j}(u,\widehat{X}^\varepsilon(u))-v_{x_j}(u,\widehat{X}(u))\right)\,d\widehat{W}_j(u)\right|^2\right]^{1/2}, \end{aligned}$$

and so,

$$\begin{aligned} &\mathbb{E}\left[\left|\int_0^{t\wedge\tau_n}\sigma_{ij}(u,\widehat{X}(u))v_{x_j}(u,\widehat{X}^\varepsilon(u))\,d\widehat{W}_j(u)-\int_0^{t\wedge\tau_n}\sigma_{ij}(u,\widehat{X}(u))v_{x_j}(u,\widehat{X}(u))\,d\widehat{W}_j(u)\right|\right]\\ &\leq\mathbb{E}\left[\int_0^{t\wedge\tau_n}|\sigma_{ij}(u,\widehat{X}(u))|^2|v_{x_j}(u,\widehat{X}^\varepsilon(u))-v_{x_j}(u,\widehat{X}(u))|^2\,du\right]^{1/2} \end{aligned} \tag{3.20}$$

Since $v_{x_j} \in C_{\mathrm{loc}}(\overline{\mathbb{H}}_T)$, we have $\mathbb{P}$-a.s., for all $0 \leq u \leq T$,

$$|\sigma_{ij}(u,\widehat{X}(u))||v_{x_j}(u,\widehat{X}^\varepsilon(u)) - v_{x_j}(u,\widehat{X}(u))| \to 0 \quad \text{ as } \varepsilon \downarrow 0.$$

By the continuity of paths of $\widehat{X}$ and the fact that σ_{ij} satisfy the growth condition (2.19), the Lebesgue Dominated Convergence Theorem implies $\mathbb{P}$-a.s.

$$\int_0^{t\wedge\tau_n} |\sigma_{ij}(u,\widehat{X}(u))|^2|v_{x_j}(u,\widehat{X}^\varepsilon(u)) - v_{x_j}(u,\widehat{X}(u))|^2\, du \to 0 \quad \text{ as } \varepsilon \downarrow 0. \tag{3.21}$$

On the closed ball of radius n in $\overline{\mathbb{H}}$ centered at the origin, the coefficients σ_{ij} and v_{x_j} are bounded, so it follows

$$\mathbb{E}\left[\int_0^{t\wedge\tau_n} |\sigma_{ij}(u,\widehat{X}(u))|^2|v_{x_j}(u,\widehat{X}^\varepsilon(u)) - v_{x_j}(u,\widehat{X}(u))|^2\, du\right]^{1/2} \to 0, \quad \text{ as } \varepsilon \downarrow 0.$$

Thus, by (3.20), we obtain the L^1-convergence, and also the $\mathbb{P}$-a.s convergence of a subsequence which we label the same as the given sequence, as $\varepsilon \downarrow 0$,

$$\int_0^{t\wedge\tau_n} \sigma_{ij}(u,\widehat{X}(u))v_{x_j}(u,\widehat{X}^\varepsilon(u))\, d\widehat{W}_j(u) \to \int_0^{t\wedge\tau_n} \sigma_{ij}(u,\widehat{X}(u))v_{x_j}(u,\widehat{X}(u))\, d\widehat{W}_j(u). \tag{3.22}$$

We write the du-integrand in (3.18) as the sum of $(\partial_t + \mathscr{A}_t)v(u,\widehat{X}^\varepsilon(u))$ and $\mathscr{R}v(u,\widehat{X}^\varepsilon(u))$, where

$$\mathscr{A}_t v(u,x^\varepsilon) = \sum_{i=1}^d b_i(u,x)v_{x_i}(u,x^\varepsilon) + \sum_{i,j=1}^d \frac{1}{2}a_{ij}(u,x)x_d^\varepsilon v_{x_ix_j}(u,x^\varepsilon), \tag{3.23}$$

$$\mathscr{R}v(u,x^\varepsilon) = -\frac{\varepsilon}{2}\sum_{i,j=1}^d a_{ij}(u,x)v_{x_ix_j}(u,x^\varepsilon), \tag{3.24}$$

for all $(u,x) \in [0,T]\times\overline{\mathbb{H}}$. An argument similar to the one which gave us (3.22) can be used to obtain the $\mathbb{P}$-a.s convergence, as $\varepsilon \downarrow 0$,

$$\int_0^{t\wedge\tau_n} (\partial_t + \mathscr{A}_t)v(u,\widehat{X}^\varepsilon(u)))\, du \to \int_0^{t\wedge\tau_n} (\partial_t + \mathscr{A}_t)v(u,\widehat{X}(u))\, du, \tag{3.25}$$

This requires that $v_t, v_{x_i}, x_d v_{x_ix_j} \in C_{\mathrm{loc}}(\overline{\mathbb{H}}_T)$, the coefficients b_i and $x_d a_{ij}$ satisfy the linear growth assumption (2.19), and coefficients a_{ij} obey (3.9). Therefore, it remains to show

$$\mathbb{E}\left[\int_0^{t\wedge\tau_n} \left|\mathscr{R}v(u,\widehat{X}^\varepsilon(u))\right|\, du\right] \to 0, \quad \text{ as } \varepsilon \downarrow 0. \tag{3.26}$$

Notice that the proof of (3.26) completes the proof of the Itô lemma because (3.19), (3.22), (3.25) and (3.26) yields (3.16) by taking limit as $\varepsilon \downarrow 0$ in (3.18).

Now, we return to the proof of (3.26). We can write the term in (3.24) in the following way

$$\begin{aligned} \varepsilon a_{ij}(u,x)v_{x_ix_j}(u,x^\varepsilon) &= \frac{\varepsilon}{x_d^\varepsilon}a_{ij}(u,x)x_d^\varepsilon v_{x_ix_j}(u,x^\varepsilon)\mathbf{1}_{\{0\leq x_d\leq\sqrt{\varepsilon}\}} \\ &\quad + \frac{\varepsilon}{x_d^\varepsilon}a_{ij}(u,x)x_d^\varepsilon v_{x_ix_j}(u,x^\varepsilon)\mathbf{1}_{\{\sqrt{\varepsilon}<x_d\}}, \quad \forall (u,x)\in\overline{\mathbb{H}}_T, \end{aligned} \tag{3.27}$$

We use the preceding identity to show the pointwise convergence, for all $(u,x) \in \overline{\mathbb{H}}_T$,

$$\varepsilon a_{ij}(u,x)v_{x_ix_j}(u,x^\varepsilon) \to 0 \quad \text{ as } \varepsilon \downarrow 0. \tag{3.28}$$

Because of the fact that a_{ij} are locally bounded on $\overline{\mathbb{H}}_T$ by (3.9) and the $x_d v_{x_ix_j} \in C_{\text{loc}}(\overline{\mathbb{H}}_T)$ obey $x_d v_{x_ix_j}(u,x) = 0$, when $x_d = 0$, we obtain

$$\frac{\varepsilon}{x_d^\varepsilon} a_{ij}(u,x) x_d^\varepsilon v_{x_ix_j}(u,x^\varepsilon) \mathbf{1}_{\{0 \leq x_d \leq \sqrt{\varepsilon}\}} \to 0, \quad \text{ as } \varepsilon \downarrow 0. \tag{3.29}$$

for all $(u,x) \in \overline{\mathbb{H}}_T$. In the case $\sqrt{\varepsilon} < x_d$, obviously we have

$$\varepsilon / x_d^\varepsilon \leq \sqrt{\varepsilon},$$

and so, using $x_d v_{x_ix_j} \in C_{\text{loc}}(\overline{\mathbb{H}}_T)$ and the local boundedness of a_{ij} on $\overline{\mathbb{H}}_T$ by (3.9) and (2.19), we obtain

$$\frac{\varepsilon}{x_d^\varepsilon} a_{ij}(u,x) x_d^\varepsilon v_{x_ix_j}(u,x^\varepsilon) \mathbf{1}_{\{\sqrt{\varepsilon} < x_d\}} \to 0, \quad \text{ as } \varepsilon \downarrow 0. \tag{3.30}$$

for all $(u,x) \in \overline{\mathbb{H}}_T$. By combining (3.29) and (3.30), we obtain (3.28). Using the continuity of the paths of $\widehat{X}$, (3.28) and (3.24), we obtain $\mathbb{P}$-a.s., for all $0 \leq u \leq T$,

$$\mathscr{R} v(u, \widehat{X}^\varepsilon(u)) \to 0, \quad \text{ as } \varepsilon \downarrow 0,$$

and also, the following holds $\mathbb{P}$-a.s.

$$\int_0^{t \wedge \tau_n} \left| \mathscr{R} v(u, \widehat{X}^\varepsilon(u)) \right| du \to 0, \quad \text{ as } \varepsilon \downarrow 0. \tag{3.31}$$

The Lebesgue Dominated Convergence Theorem, conditions (3.9) and (2.19) satisfied by a_{ij} on $\overline{\mathbb{H}}_T$, and $x_d v_{x_ix_j} \in C_{\text{loc}}(\overline{\mathbb{H}}_T)$ now imply (3.26). This concludes the proof of the proposition. □

The next result is based on the *existence* of a solution in $\mathscr{C}^{2+\alpha}(\overline{\mathbb{H}}_T)$ to the homogeneous initial value problem considered in Theorem 2.5.

Proposition 3.5 (Uniqueness of the one-dimensional marginal distributions)**.** *Assume the hypotheses of Theorem 1.10 hold. Let $(\widehat{X}^k, \widehat{W}^k)$, defined on filtered probability spaces $(\Omega^k, \mathbb{P}^k, \mathscr{F}^k)$, $\{\mathscr{F}_t^k\}_{t \geq 0}$, $k = 1, 2$, be two weak solutions to (1.2) with initial condition $(s,x) \in [0,\infty) \times \overline{\mathbb{H}}$. Then the one-dimensional marginal probability distributions of $\widehat{X}^1(t)$ and $\widehat{X}^2(t)$ agree for each $t \geq s$.*

Proof. We apply a duality argument as in the proof of [26, Lemma 5.4.26] with the aid of Theorem 2.5 and Proposition 3.4 but indicate the differences in the proof provided here. Without loss of generality, we may assume that $s = 0$. By Proposition 3.1, it is enough to show that for any $T > 0$ and $g \in C_0^\infty(\overline{\mathbb{H}})$, we have

$$\mathbb{E}_{\mathbb{P}^1} \left[g(\widehat{X}^1(T)) \right] = \mathbb{E}_{\mathbb{P}^2} \left[g(\widehat{X}^2(T)) \right], \tag{3.32}$$

where each expectation is taken under the law of the corresponding process. For this purpose, we consider the parabolic differential operator,

$$-\check{L} w(t,x) := -w_t(t,x) + \sum_{i=1}^d b_i(T-t,x) w_{x_i}(t,x) + \sum_{i,j=1}^d \frac{1}{2} x_d a_{ij}(T-t,x) w_{x_ix_j}(t,x), \tag{3.33}$$

for all $(t,x) \in \mathbb{H}_T$ and $w \in C^{1,2}(\mathbb{H}_T)$. Let $u \in \mathscr{C}^{2+\alpha}(\overline{\mathbb{H}}_T)$ be the unique solution given by Theorem 2.5 to the homogeneous initial value problem,

$$\begin{cases} \check{L}(t,x) = 0, & \text{for } (t,x) \in (0,T) \times \mathbb{H}, \\ u(0,x) = g(x), & \text{for } x \in \overline{\mathbb{H}}. \end{cases} \tag{3.34}$$

Define

$$v(t,x) := u(T-t,x), \quad \forall (t,x) \in [0,T] \times \overline{\mathbb{H}}. \tag{3.35}$$

Then, $v \in \mathscr{C}^{2+\alpha}(\overline{\mathbb{H}}_T)$ solves the terminal value problem,

$$\begin{cases} v_t(t,x) + \mathscr{A}_t v(t,x) = 0, & \text{for } (t,x) \in (0,T) \times \mathbb{H}, \\ v(T,x) = g(x), & \text{for } x \in \overline{\mathbb{H}}, \end{cases} \tag{3.36}$$

where the differential operator $\mathscr{A}_t$ is given by (1.1). Proposition 3.4 gives us, for $k = 1,2$,

$$\begin{aligned} \mathbb{E}_{\mathbb{P}^k}\left[v(T,\widehat{X}^k(T))\right] &= v(0,x) + \mathbb{E}_{\mathbb{P}^k}\left[\int_0^T (v_t + \mathscr{A}_t)\, v(t,\widehat{X}^k(t))\, dt\right] \\ &\quad + \mathbb{E}_{\mathbb{P}^k}\left[\int_0^T \sum_{i,j=1}^d \sigma_{ij}(t,\widehat{X}^k(t)) v_{x_j}(t,\widehat{X}^k(t))\, d\widehat{W}_j^k(t)\right]. \end{aligned} \tag{3.37}$$

Recall that $v_{x_i} \in C(\overline{\mathbb{H}}_T)$ and the coefficients σ_{ij} satisfy (2.19). Inequality (3.13) applied with $m = 1$, gives

$$\mathbb{E}_{\mathbb{P}^k}\left[\int_0^T \left|\sigma_{ij}(t,\widehat{X}^k(t)) v_{x_j}(t,\widehat{X}^k(t))\right|^2 dt\right] \leq C(1+|x|^2)\|v_{x_i}\|^2_{C(\overline{\mathbb{H}}_T)},$$

and so, the Itô integrals in (3.37) are square-integrable, continuous martingales, which implies

$$\mathbb{E}_{\mathbb{P}^k}\left[\int_0^T \sigma_{ij}(t,\widehat{X}^k(t)) v_{x_j}(t,\widehat{X}^k(t))\, d\widehat{W}_j^k(t)\right] = 0.$$

Using the preceding inequality and (3.36), we see that (3.37) yields

$$\mathbb{E}_{\mathbb{P}^k}\left[g(\widehat{X}^k(T))\right] = v(0,x), \quad k = 1,2, \tag{3.38}$$

and so, (3.32) follows. □

Next, we recall

Proposition 3.6 (Uniqueness of solutions to the classical martingale problem)**.** [26, Proposition 5.4.27] [19, Theorem 4.4.2 & Corollary 4.4.3]. *Let*

$$\begin{aligned} \tilde{b} &: \mathbb{R}^d \to \mathbb{R}^d, \\ \tilde{\sigma} &: \mathbb{R}^d \to \mathbb{R}^{d\times d}, \end{aligned}$$

be Borel measurable functions such that they are bounded on each compact subset in $\mathbb{R}^d$ and define a differential operator by

$$\mathscr{G}u(x) := \sum_{i=1}^d \tilde{b}_i(x) u_{x_i} + \sum_{i,j=1}^d \frac{1}{2}\tilde{a}_{ij}(x) u_{x_i x_j}, \quad \forall x \in \mathbb{R}^d,$$

where $\tilde{a} := \tilde{\sigma}\tilde{\sigma}^*$ and $u \in C^2(\mathbb{R}^d)$. Suppose that for every $x \in \mathbb{R}^d$, any two solutions $\mathbb{P}^x$ and $\mathbb{Q}^x$ to the time-homogeneous martingale problem associated with $\mathscr{G}$ have the same one-dimensional marginal distributions. Then, for every initial condition $x \in \mathbb{R}^d$, there exists at most one solution to the time-homogeneous martingale problem associated to $\mathscr{G}$.

We have the following consequence of Propositions 3.5 and 3.6; the difference between [26, Proposition 5.4.27] and our Corollary 3.7 is that the coefficients in [26, Proposition 5.4.27] do not depend on time, while our coefficients do depend on time. The proof of Corollary 3.7 proceeds by reducing the problem to one which Proposition 3.6 may be applied.

Corollary 3.7 (Uniqueness of solutions to the martingale problem associated to $\mathscr{A}_t$)**.** *Suppose that for every $x \in \overline{\mathbb{H}}$ and $s \geq 0$, any two solutions $\mathbb{P}^{s,x}$ and $\mathbb{Q}^{s,x}$ to the martingale problem in Definition 1.1 associated to $\mathscr{A}_t$ in* (1.1) *with initial condition (s,x) have the same one-dimensional marginal distributions. Then, for every initial condition $(s,x) \in [0,\infty) \times \overline{\mathbb{H}}$, there exists at most one solution to the martingale problem associated to $\mathscr{A}_t$.*

Proof. As (1.2) is time-inhomogeneous with initial condition $(s,x) \in [0,\infty) \times \overline{\mathbb{H}}$, rather than time-homogeneous with initial condition $x \in \mathbb{R}^d$, as assumed by Proposition 3.6, we first extend the coefficients, $\sigma(t,x)$ and $b(t,x)$ with $(t,x) \in [0,\infty) \times \overline{\mathbb{H}}$ to $(t,x) \in \mathbb{R} \times \mathbb{R}^{d-1} \times (-\infty, 0)$ and $(t,x) \in (-\infty,0) \times \overline{\mathbb{H}}$, so

$$\tilde{\sigma}_{ij}(t,x) = 0, \quad \tilde{b}_i(t,x) = 0, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^{d-1} \times (-\infty,0) \text{ and } (t,x) \in (-\infty,0) \times \overline{\mathbb{H}}. \tag{3.39}$$

To obtain a time-homogeneous differential operator, as in Proposition 3.6, we increase the space dimension by adding the time coordinate, that is, we consider the following $(d+1)$-dimensional process

$$\begin{aligned} dY_0(t) &= dt, \quad \forall t \geq 0, \\ dY_i(t) &= \tilde{b}_i(Y(t))dt + \sum_{j=1}^d \tilde{\sigma}_{ij}(Y(t))dW_j(t), \quad i = 1,\dots,d, \quad \forall t \geq 0. \end{aligned} \tag{3.40}$$

Now, let $\mathscr{G}$ denote the time-homogeneous differential operator

$$\mathscr{G}u(y) := \sum_{i=1}^d \tilde{b}_i(y) u_{y_i} + \sum_{i,j=1}^d \frac{1}{2}\tilde{a}_{ij}(y) u_{y_i y_j}, \quad \forall y \in \mathbb{R}^{d+1},$$

and $u \in C^2(\mathbb{R}^{d+1})$.

For $x \in \overline{\mathbb{H}}$ and $s \geq 0$, let $\mathbb{P}^{s,x}$ and $\mathbb{Q}^{s,x}$ be two solutions to the martingale problem associated to $\mathscr{A}_t$ with initial condition (s,x). We extend these probability measures from the measurable space $C_{\text{loc}}([0,\infty); \overline{\mathbb{H}})$ to the canonical space $C_{\text{loc}}([0,\infty); \mathbb{R}^d)$ in the following way

$$\begin{aligned} &\widetilde{\mathbb{P}}^{s,x}\left(\widetilde{\omega} \in C_{\text{loc}}([0,\infty);\mathbb{R}^{d+1}), \exists t \geq 0, \widetilde{\omega}_0(t) \neq t+s\right) = 0, \\ &\widetilde{\mathbb{P}}^{s,x}\left(\widetilde{\omega} \in C_{\text{loc}}([0,\infty);\mathbb{R}^{d+1}), (\widetilde{\omega}_1(t_i),\dots,\widetilde{\omega}_d(t_i)) \in B_i, i = 1,\dots,m\right) \\ &\qquad = \mathbb{P}^{s,x}\left(\omega \in C_{\text{loc}}([0,\infty);\overline{\mathbb{H}}), (\omega_1(t_i),\dots,\omega_d(t_i)) \in B_i, i=1,\dots,m\right), \end{aligned}$$

for all $m \geq 1$, $0 \leq t_1 \leq t_2 \leq \ldots \leq t_m$, $B_i \in \mathscr{B}(\mathbb{R}^d)$, $i = 1,\dots,m$. Above, we used the notation

$$\begin{aligned} \tilde{\omega} &:= (\tilde{\omega}_0, \tilde{\omega}_1, \dots, \tilde{\omega}_d), \quad \forall \tilde{\omega} \in C_{\text{loc}}([0,\infty);\mathbb{R}^{d+1}), \\ \omega &:= (\omega_1,\dots,\omega_d), \quad \forall \omega \in C_{\text{loc}}([0,\infty);\overline{\mathbb{H}}). \end{aligned}$$

Similarly, we build $\widetilde{\mathbb{Q}}^{s,x}$ the extension of $\mathbb{Q}^{s,x}$ from $C_{\text{loc}}([0,\infty);\overline{\mathbb{H}})$ to $C_{\text{loc}}([0,\infty);\mathbb{R}^d)$. Notice that $\widetilde{\mathbb{P}}^{s,x}$ and $\widetilde{\mathbb{Q}}^{s,x}$ are two solutions to the classical time-homogeneous martingale problem associated to $\mathscr{G}$, with initial condition (s,x). Therefore, the probability measures $\mathbb{P}^{s,x}$ and $\mathbb{Q}^{s,x}$ coincide if their extensions $\widetilde{\mathbb{P}}^{s,x}$ and $\widetilde{\mathbb{Q}}^{s,x}$ coincide. By Proposition 3.6, uniqueness in law holds for $\widetilde{\mathbb{P}}^{s,x}$ and $\widetilde{\mathbb{Q}}^{s,x}$ if, for any $y = (y_0,\dots,y_d) \in \mathbb{R}^{d+1}$ and any two solutions $\widetilde{\mathbb{P}}^y_i$, $i = 1,2$, to the classical

martingale problem associated to $\mathscr{G}$ with initial condition y, their one-dimensional marginal distribution coincide. For $i = 1, 2$, let Y^i be the weak solution to (3.40) with initial condition $Y^i(0) = y$ such that the law of Y^i is given by $\widetilde{\mathbb{P}}_i^y$ (see [26, Proposition 5.4.6 & Corollary 5.4.8]). Then, the one-dimensional marginal distributions agree for the probability measure $\widetilde{\mathbb{P}}_i^y$, $i = 1, 2$, if and only if they agree for the stochastic processes Y^i, $i = 1, 2$. Next, we show that uniqueness of the one-dimensional marginal distributions of Y^i, $i = 1, 2$, holds. For this purpose, we consider two cases.

Case 1 ($y_d < 0$ or $y_0 < 0$). In this case, the coefficients $\tilde{b}$ and $\tilde{\sigma}$ are identically zero on a neighborhood of y, and so the unique solution, Y, to (3.40) is given by $Y(t) = y$, for all $t \geq 0$. It is obvious that the one-dimensional marginal distributions of solutions Y^i, $i = 1, 2$, to (3.40) are uniquely determined in this situation.

Therefore, by Proposition 3.6, uniqueness in law holds for solutions to (3.40) if the one-dimensional marginal distributions are uniquely determined for any initial condition $y \in \mathbb{R}^{d+1}$ with $y_d \geq 0$ and $y_0 \geq 0$.

Case 2 ($y_d \geq 0$ and $y_0 \geq 0$). Note that any weak solution, $(Y(t))_{t\geq 0}$, to (3.40) with initial condition $Y(0) = y$, satisfies the property that

$$Y_d(t) \geq 0 \quad \mathbb{P}\text{-a.s.}, \quad \forall t \geq 0. \tag{3.41}$$

If this were not so, then there would be an $\varepsilon > 0$ such that Y_d reached the level $-\varepsilon$ with non-zero probability. By the preceding case, we observe that the Y_d would remain at the level $-\varepsilon$ for any subsequent time. By the continuity of paths, Y_d would have hit $-\varepsilon/2$ at a preceding time, and again, the preceding case would imply that Y_d remained at $-\varepsilon/2$ for all subsequent times. But this would contradict our assumption and therefore, (3.41) holds.

Any weak solution, $(Y(t))_{t\geq 0}$, to (3.40) with initial condition $Y(0) = y$ gives a solution, $(\widehat{X}(t))_{t\geq y_0}$,

$$\widehat{X}(t) = (Y_1(t - y_0), Y_2(t - y_0), \dots, Y_d(t - y_0)) \quad \forall t \geq y_0, \tag{3.42}$$

to the stochastic differential equation

$$d\widehat{X}_i(t) = \widetilde{b}_i(t, \widehat{X}(t))dt + \sum_{j=1}^{d} \widetilde{\sigma}_{ij}(t, \widehat{X}(t))dW_j(t), \quad i = 1, \dots, d, \quad \forall t \geq y_0,$$

with initial condition

$$\widehat{X}(y_0) = (Y_1(0), \dots, Y_d(0)) = (y_1, \dots, y_d) \in \overline{\mathbb{H}}.$$

Moreover, X remains in $\overline{\mathbb{H}}$, for all $t \geq y_0$, by (3.41).

Therefore, the one-dimensional marginal distributions of Y are uniquely determined if the marginal distributions of $\widehat{X}$ are uniquely determined. But, the last statement is implied if the one-dimensional marginal distributions of any solution $\mathbb{P}^{s,x}$ to the martingale problem associated to $\mathscr{A}_t$, with initial condition $(s, x) \in [0, \infty) \times \overline{\mathbb{H}}$, are uniquely determined.

Combining the conclusions of the preceding two cases completes the proof of the corollary.

□

Finally, we have

Proof of Theorem 1.3. The result follows from Theorem 3.3 which asserts the existence of solutions to the martingale problem associated to $\mathscr{A}_t$, while Proposition 3.5 and Corollary 3.7 show

that the solution is unique. Therefore, the martingale problem associated to $\mathscr{A}_t$ is well-posed, for any initial condition $(s,x) \in [0,\infty) \times \overline{\mathbb{H}}$. □

Proof of Theorem 1.10. By Theorem 3.3, we obtain existence of weak solutions to (1.2). Since each weak solution induces a probability measure on $C_{\text{loc}}([0,\infty);\overline{\mathbb{H}})$ which solves the martingale problem associated to $\mathscr{A}_t$, we obtain by Theorem 1.3 that the probability law of the weak solutions to (1.2) is uniquely determined.

The fact that the weak solutions to (1.2) satisfy the strong Markov property can be shown to follow by the same argument applied in the time-homogeneous case in [19, Theorem 4.4.2 (b) & (c)]; an alternative argument is provided below.

To prove the strong Markov property of weak solutions to (1.2), we consider again the time-homogeneous stochastic differential equation (3.40) from the proof of Corollary 3.7. The same argument as the one used to conclude that the martingale problem associated to $\mathscr{A}_t$ is well-posed can be used to conclude that the classical martingale problem associated to the stochastic differential equation (3.40) is well-posed. Therefore, by [26, Theorem 5.4.20], we obtain that for any $y \in \mathbb{R}^{d+1}$, the weak solution Y^y to (3.40) started at y possesses the strong Markov property, that is for any stopping time T of $\{\mathscr{B}_t(C_{\text{loc}}([0,\infty);\mathbb{R}^{d+1}))\}_{t\geq 0}$, any Borel measurable set $B \in \mathscr{B}(\mathbb{R}^{d+1})$ and $u \geq 0$, we have

$$\widetilde{\mathbb{P}}^y(Y(T+u) \in B | \mathscr{B}_T(C_{\text{loc}}([0,\infty);\mathbb{R}^{d+1})) = \widetilde{\mathbb{P}}^y(Y(T+u) \in B | Y(T)), \tag{3.43}$$

where $\widetilde{\mathbb{P}}^y$ denotes the probability law of the process Y started at y. Let $(s,x) \in [0,\infty) \times \overline{\mathbb{H}}$ and let $\widehat{X}^{s,x}$ be the unique weak solution of (1.2) with initial condition $\widehat{X}^{s,x}(s) = x$. Let $\mathbb{P}^{s,x}$ denote the probability law of $\widehat{X}^{s,x}$. Then, by analogy with (3.42), we notice that

$$Y^{s,x}(t) := \Big(t+s, \widehat{X}_1(t+s), \ldots, \widehat{X}_d(t+s)\Big) \quad t \geq 0,$$

is a solution to (3.40) with initial condition (s,x). Therefore, (3.43) can be rewritten in terms of the probability law of $\widehat{X}^{s,x}$, $\mathbb{P}^{s,x}$, as follows

$$\mathbb{P}^{s,x}(\widehat{X}(T+u) \in B | \mathscr{B}_T(C_{\text{loc}}([0,\infty);\overline{\mathbb{H}})) = \mathbb{P}^{s,x}(\widehat{X}(T+u) \in B | X(T)), \tag{3.44}$$

for any stopping time T of $(\mathscr{B}_t(C_{\text{loc}}([0,\infty);\overline{\mathbb{H}})))_{t\geq 0}$, any Borel measurable set $B \in \mathscr{B}(\overline{\mathbb{H}}))$ and $u \geq s$. Thus, $\widehat{X}^{s,x}$ satisfies the strong Markov property. □

3.3. **Matching one-dimensional marginal probability distributions.** We can now complete the proof of Theorem 1.12. For simplicity, we denote

$$\alpha(t) := \xi(t)\xi^*(t), \quad \forall t \geq 0.$$

First, we prove the analogue of Proposition 3.1 for the Itô process (1.3).

Proposition 3.8. *Let X be the Itô process* (1.3)*, such that $X(0) \in \overline{\mathbb{H}}$. Assume the coefficients σ and b_d defined by* (1.16) *and* (1.17) *(now defined on $[0,\infty) \times \mathbb{R}^d$) satisfy* (3.1) *and* (3.2)*, respectively. Then*

$$\mathbb{P}\left(X(t) \in \overline{\mathbb{H}} | X(0)\right) = 1, \quad \forall t \geq 0. \tag{3.45}$$

Proof. The argument is similar to the proof of Proposition 3.1. We include it for completeness. It suffices to show that, for any $\varepsilon > 0$, we have

$$\mathbb{P}(X_d(t) \in (-\infty, -\varepsilon)) = 0. \tag{3.46}$$

Let $\varphi : \mathbb{R} \to [0,1]$ be the smooth cut-off function defined in the proof Proposition 3.1. The Itô lemma gives

$$\begin{aligned}\varphi(X_d(t)) &= \varphi(X_d(0)) + \int_0^t \left[\beta_d(s)\varphi'(X_d(s)) + \frac{1}{2}\alpha_{dd}(s)\varphi''(X_d(s))\right] ds\\ &\quad + \int_0^t \sigma_{d,i}(s)\varphi'(X_d(s))\, dW_i(s).\end{aligned}$$

By taking conditional expectations in the preceding expression and using the fact that the last term in that expression is a martingale by (1.13), we obtain

$$\begin{aligned}\mathbb{E}\left[\varphi(X_d(t))\right] &= \mathbb{E}\left[\varphi(X_d(0))\right] + \mathbb{E}\left[\int_0^t \left(\beta_d(s)\varphi'(X_d(s)) + \frac{1}{2}\alpha_{dd}(s)\varphi''(X_d(s))\right) ds\right]\\ &= \mathbb{E}\left[\varphi(X_d(0))\right] + \int_0^t \mathbb{E}\Big[\mathbb{E}\left[\beta_d(s)|X(s)\right]\varphi'(X_d(s))\\ &\quad + \frac{1}{2}\mathbb{E}\left[\alpha_{dd}(s)|X(s)\right]\varphi''(X_d(s))\Big]\, ds\\ &= \mathbb{E}\left[\varphi(X_d(0))\right] + \mathbb{E}\Big[\int_0^t \Big(b_d(s,X(s))\varphi'(X_d(s))\\ &\quad + \frac{1}{2}X_d^+(s)a_{dd}(s,X(s))\varphi''(X_d(s))\Big)\, ds\Big].\end{aligned}$$

We have $b_d(s,X(s))\varphi'(X(s)) \leq 0$ and $\varphi(X_d(0)) = 0$, while $\varphi(X_d(t)) \geq 0$. Therefore, $\mathbb{E}\left[\varphi(X_d(t))\right] \leq 0$ and thus $\mathbb{E}\left[\varphi(X_d(t))\right] = 0$, which yields (3.46). □

Next, we have

Proof of Theorem 1.12. Let $\widehat{X}$ be the unique weak solution to the mimicking stochastic differential equation (1.2) with initial condition $\widehat{X}(0) = X(0) = x$. As in the proof of Proposition 3.5, we need to show that for any $T \geq 0$ and $g \in C_0^\infty(\overline{\mathbb{H}})$, we have

$$\mathbb{E}\left[g(\widehat{X}(T))\right] = \mathbb{E}\left[g(X(T))\right]. \tag{3.47}$$

Let $v \in \mathscr{C}^{2+\alpha}(\overline{\mathbb{H}}_T)$ be defined by (3.33), (3.34) and (3.35). Then, (3.38) gives

$$\mathbb{E}\left[g(\widehat{X}(T))\right] = v(0,x). \tag{3.48}$$

We wish to prove that (3.48) holds with $X(T)$ in place of $\widehat{X}(T)$. We proceed as in the proof of Proposition 3.5. By applying the Itô lemma to $v(t,X^\varepsilon(t))$, we obtain

$$\begin{aligned}dv(t,X^\varepsilon(t)) &= \left(v_t(t,X^\varepsilon(t)) + \sum_{i=1}^d \beta_i(t)v_{x_i}(t,X^\varepsilon(t)) + \sum_{i,j=1}^d \frac{1}{2}\alpha_{ij}(t)v_{x_ix_j}(t,X^\varepsilon(t))\right) dt\\ &\quad + \sum_{i,j=1}^d \xi_{ij}(t)v_{x_i}(t,X^\varepsilon(t))dW_j(t).\end{aligned}$$

The $dW_j(t)$-terms in the preceding identity are square-integrable, continuous martingales, because

$$[0,T] \ni t \mapsto v_{x_i}(t,X^\varepsilon(t))$$

are bounded processes since $v_{x_i} \in C([0,T] \times \overline{\mathbb{H}})$, and $\xi(t)$ is a square-integrable, adapted process by (1.13). Therefore,

$$\begin{aligned}\mathbb{E}\left[v(T, X^\varepsilon(T))\right] &= v(0, x^\varepsilon)\\ &+ \mathbb{E}\left[\int_0^T \left(v_t(t, X^\varepsilon(t)) + \sum_{i=1}^d \beta_i(t) v_{x_i}(t, X^\varepsilon(t)) + \sum_{i,j=1}^d \frac{1}{2}\alpha_{ij}(t) v_{x_i x_j}(t, X^\varepsilon(t))\right) dt\right].\end{aligned}$$

Using conditional expectations, we may rewrite the preceding identity as

$$\begin{aligned}\mathbb{E}\left[v(T, X^\varepsilon(T))\right] &= v(0, x^\varepsilon)\\ &+ \int_0^T \mathbb{E}\left[\left(v_t(t, X^\varepsilon(t)) + \sum_{i=1}^d \beta_i(t) v_{x_i}(t, X^\varepsilon(t)) + \sum_{i,j=1}^d \frac{1}{2}\alpha_{ij}(t) v_{x_i x_j}(t, X^\varepsilon(t))\right) \middle| X^\varepsilon(t)\right] dt\\ &= v(0, x^\varepsilon) + \mathbb{E}\left[\int_0^T \left(v_t(t, X^\varepsilon(t)) + \mathscr{A}_t v(t, X^\varepsilon(t))\right) dt\right].\end{aligned}$$

Since $v_t(t,x) + \mathscr{A}_t v(t,x) = 0$, for all $(t,x) \in \mathbb{H}_T$, by letting $\varepsilon \downarrow 0$ in the preceding identity, we obtain

$$\mathbb{E}\left[g(X(T))\right] = v(0,x),$$

and this concludes the proof by (3.48). □

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(PF) Department of Mathematics, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019
E-mail address, PF: `feehan@math.rutgers.edu`

(CP) Department of Mathematics, University of Pennsylvania, 209 South 33rd Street, Philadelphia, PA 19104-6395
E-mail address: `cpop@math.upenn.edu`